STEP 2010, Paper 2, Q4 – Solution (2 pages; 9/6/18)

Introduction

The first part of the question regularly involves something very obvious (though in this case, a slight modification is needed!)

Having established the initial result, $\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)} dx$ is most likely to be a straightforward application of it.

In the case of definite integrals, the limits can be very revealing. Note that $\int_a^b = -\int_b^a$

For the subsequent integrals, there is bound to be some complication. Possibilities include:

(a) Some manipulation may be needed in order to be able to apply the result.

(b) Generalising the earlier result in some way [eg if it had involved $\int sinxcos^3xdx$, this could be generalised to $\int sinxcos^nxdx$]

(c) Deriving a new result by applying a similar idea.

Try observing any interesting features of the question. Does anything stand out in the case of $\int_{1/2}^2 \frac{sinx}{x(sinx+sin\left(\frac{1}{x}\right))} dx$?

Solution

(i) Let $x = a - u$ [the u will then be replaced with x, to give $f(a-x)$ in the numerator], so that

$I = \int_a^0 \frac{-f(a-u)}{f(a-u)+f(u)} du = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)} dx$, giving the required result.

Then $2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)} dx = \int_0^a dx = a$, so that $I = \frac{a}{2}$

$I_1 = \int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}dx = \int_0^1 \frac{\ln(x+1)}{\ln[(2-x)(1+x)]}dx = \int_0^1 \frac{\ln(x+1)}{\ln(2-x)+ln(1+x)}dx$

Using the previous result, a will be 1, and $\ln(x+1)$ will be $f(x)$, making $f(a-x) = \ln([1-x]+1) = \ln(2-x)$,

so that $I_1 = 1/2$

$I_2 = \int_0^{\frac{\pi}{2}} \frac{sinx}{\sin(x+\frac{\pi}{4})}dx = \int_0^{\frac{\pi}{2}} \frac{sinx}{\frac{1}{\sqrt{2}}(sinx+cosx)}dx = \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{sinx}{sinx+\sin(\frac{\pi}{2}-x)}dx$

$= \sqrt{2}\left(\frac{\left(\frac{\pi}{2}\right)}{2}\right) = \frac{\pi\sqrt{2}}{4}$

(ii) In the case of $\int_{1/2}^{2} \frac{sinx}{x(sinx+sin\left(\frac{1}{x}\right))}dx$, the bottom limit isn't 0, and there isn't an obvious substitution that produces 0 ($u = x - \frac{1}{2}$ doesn't look very promising, for example), but it may be possible to apply a similar idea to that used to produce the original result.

Let $u = 1/x$, so that $du = -(1/x^2)dx$, so that $\frac{dx}{x} = -\frac{du}{u}$

Then $I_3 = \int_{1/2}^{2} \frac{sinx}{x(sinx+sin\left(\frac{1}{x}\right))}dx = \int_{2}^{1/2} \frac{-sin\left(\frac{1}{u}\right)}{u(sin\left(\frac{1}{u}\right)+sinu)}du =$

$\int_{1/2}^{2} \frac{sin\left(\frac{1}{x}\right)}{x(sin\left(\frac{1}{x}\right)+sinx)}dx$

Hence $2I_3 = \int_{\frac{1}{2}}^{2} \frac{sinx+sin\left(\frac{1}{x}\right)}{x(sinx+sin\left(\frac{1}{x}\right))}dx = \int_{\frac{1}{2}}^{2} \frac{1}{x}dx = ln2 - \ln\left(\frac{1}{2}\right)$

$= ln2 + ln2 = 2ln2$, so that $I_3 = ln2$