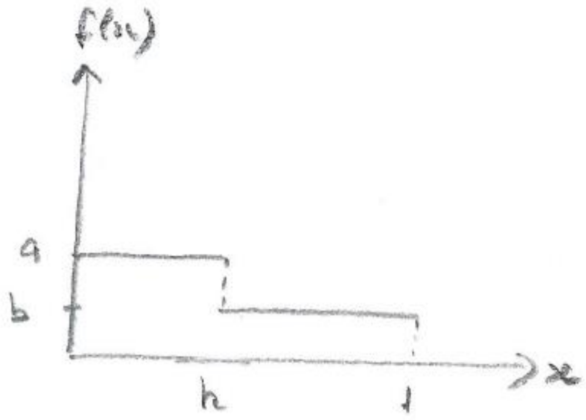
STEP 2010, Paper 2, Q12 – Solution (3 pages; 9/6/18)



$Prob(0 \leq X \leq 1) = 1$, so that $ak + b(1-k) = 1$

$\Rightarrow k(a-b) = 1 - b \Rightarrow k = \frac{1-b}{a-b}$ (1)

Then, as $k > 0$ & $a > b$, it follows that $1 - b > 0$; ie $b < 1$

Also (1)$\Rightarrow ka - kb = 1 - b$

$\Rightarrow a = \frac{1-b+kb}{k} = 1 + \frac{(1-b+kb)-k}{k}$

$= 1 + \frac{(1-k)(1-b)}{k} > 1$, as $k < 1, b < 1$ & $k > 0$

(i) $E(X) = \int_0^k xa\,dx + \int_k^1 xb\,dx$

$= a\left[\frac{1}{2}x^2\right]_0^k + b\left[\frac{1}{2}x^2\right]_k^1$

$= \frac{a}{2}k^2 + \frac{b}{2}(1-k^2)$

$= k^2\left(\frac{a}{2} - \frac{b}{2}\right) + \frac{b}{2} = \frac{1}{2}\left(\frac{1-b}{a-b}\right)^2(a-b) + \frac{b}{2}$

$= \frac{1}{2(a-b)}\{(1-b)^2 + b(a-b)\}$

$= \frac{1}{2(a-b)}\{1 - 2b + ab\}$, as required

(ii) If $0 < M \leq k$, then $P(X \leq k) \geq \frac{1}{2}$

and hence $ka \geq \frac{1}{2}$, so that $\left(\frac{1-b}{a-b}\right) a \geq \frac{1}{2}$

and $2a - 2ab \geq a - b$, giving $a + b \geq 2ab$

Then $P(X \leq M) = \frac{1}{2} \Rightarrow Ma = \frac{1}{2}$ and hence $M = \frac{1}{2a}$

If $M \geq k$ (so that $ka \leq \frac{1}{2}$ and hence $a + b \leq 2ab$), then

$P(X \geq M) = \frac{1}{2} \Rightarrow b(1 - M) = \frac{1}{2}$

$\Rightarrow 1 - M = \frac{1}{2b}$ and $M = 1 - \frac{1}{2b}$

(iii) **Case 1:** $\boldsymbol{0 < M \leq k}$

$E(X) - M = \frac{1}{2(a-b)}\{1 - 2b + ab\} - \frac{1}{2a}$

$= \frac{1}{2a(a-b)}\{a(1 - 2b + ab) - (a - b)\}$

$= \frac{b}{2a(a-b)}\{-2a + a^2 + 1) = \frac{b(1-a)^2}{2a(a-b)} > 0$,

as $b > 0, a \neq 1, a > 0$ & $a > b$

Thus $E(X) > M$

Case 2: $\boldsymbol{M > k}$

$E(X) - M = \frac{1}{2(a-b)}\{1 - 2b + ab\} - (1 - \frac{1}{2b})$

$= \frac{1}{2b(a-b)}\{b(1 - 2b + ab) - 2b(a - b) + a - b\}$

$$= \frac{(b-2b^2+ab^2-2ab+2b^2+a-b)}{2b(a-b)}$$

$$= \frac{a(b^2-2b+1)}{2b(a-b)} = \frac{a(b-1)^2}{2b(a-b)} > 0, \quad \text{as } a > 0, b \neq 1, b > 0 \ \&\ a > b$$

Thus $E(X) > M$ again.