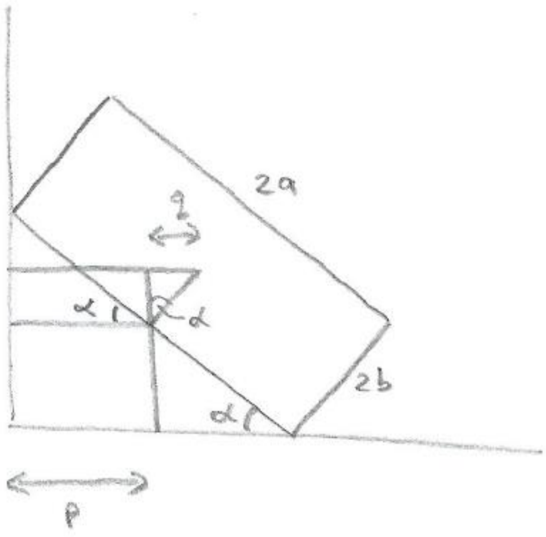
STEP 2010, Paper 1, Q9 – Solution (3 pages; 8/6/18)

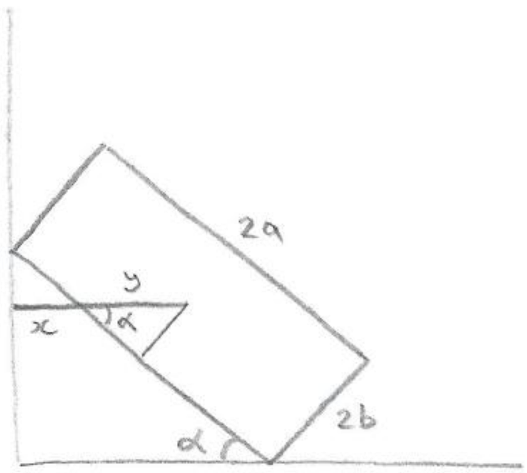
To find the distance of the centre of mass from the wall:

Method 1



$p = a\cos\alpha$; $q = b\sin\alpha$, and $p + q = a\cos\alpha + b\sin\alpha$, as required

Method 2



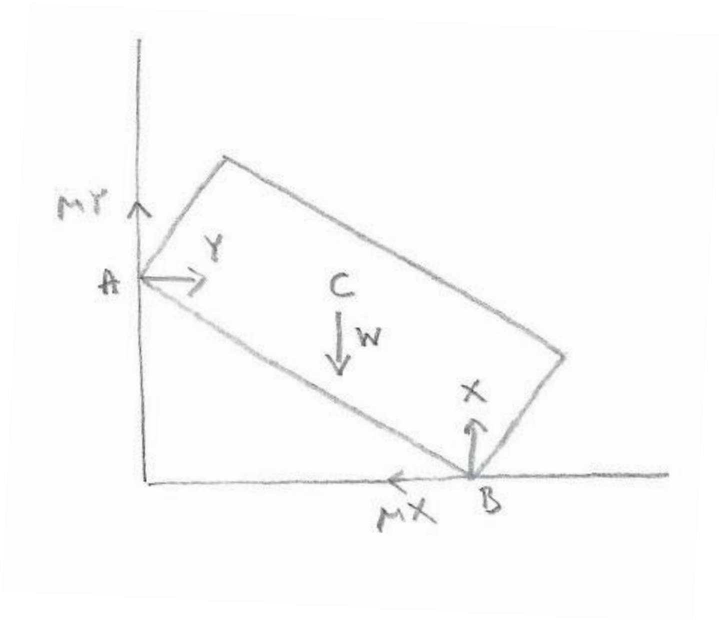
$y = \frac{b}{\sin\alpha}$; $x = \left(a - \frac{b}{\tan\alpha}\right)\cos\alpha$

so that $x + y = a\cos\alpha + b\left(\frac{1}{\sin\alpha} - \frac{\cos\alpha}{\tan\alpha}\right)$

$= acos\alpha + bsin\alpha\left(\frac{1}{sin^2\alpha} - \frac{cos\alpha}{sin\alpha tan\alpha}\right)$

Then $\frac{1}{sin^2\alpha} - \frac{cos\alpha}{sin\alpha tan\alpha} = \frac{1}{sin^2\alpha}\left(1 - \frac{sin^2\alpha cos\alpha}{sin\alpha tan\alpha}\right) = \frac{1}{sin^2\alpha}(1 - cos^2\alpha) = 1$

giving $x + y = acos\alpha + bsin\alpha$



[It is tempting to take moments about C, in order to exclude W from the equations. However, it is easier to take moments about A or B, as W can easily be substituted for.]

$M(A): X(2acos\alpha) - \mu X(2asin\alpha) - W(acos\alpha + bsin\alpha) = 0$ (1)

Resolving vert: $\mu Y + X = W$ (2)

Resolving horiz: $Y = \mu X$ (3)

Subst. for W & Y into (1):

$X(2acos\alpha) - \mu X(2asin\alpha) - (\mu^2 X + X)(acos\alpha + bsin\alpha) = 0$

$\Rightarrow 2cos\alpha - 2tan\lambda sin\alpha - sec^2\lambda\left(cos\alpha + \frac{bsin\alpha}{a}\right) = 0$

$\Rightarrow cos\alpha + \frac{bsin\alpha}{a} = 2cos^2\lambda(cos\alpha - tan\lambda sin\alpha)$

$\Rightarrow \frac{bsin\alpha}{a} = cos\alpha(2cos^2\lambda - 1) - 2sin\lambda cos\lambda sin\alpha$

$= cos\alpha cos2\lambda - sin2\lambda sin\alpha$

$= \cos(2\lambda + \alpha)$

so that $a\cos(2\lambda + \alpha) = bsin\alpha$, as required

If the lamina is square, $\cos(2\lambda + \alpha) = sin\alpha = cos\left(\frac{\pi}{2} - \alpha\right)$,

so that (a) $2\lambda + \alpha = \frac{\pi}{2} - \alpha$ or (possibly) (b) $\frac{\pi}{2} - \alpha + 2\pi$

$(a) \Rightarrow \lambda = \frac{1}{2}\left(\frac{\pi}{2} - 2\alpha\right) = \frac{\pi}{4} - \alpha$

$(b) \Rightarrow \lambda = \frac{1}{2}\left(\frac{\pi}{2} - 2\alpha + 2\pi\right) = \frac{\pi}{4} - \alpha + \pi = \left(\frac{\pi}{2} - \alpha\right) + \frac{3\pi}{4}$

Assuming that $0 < \lambda < \frac{\pi}{2}$ ensures that (b) does not apply, since $\frac{\pi}{2} - \alpha > 0$

[In theory, there is nothing to stop $\lambda = \frac{\pi}{6} + \pi$, for example, but $0 < \lambda < \frac{\pi}{2}$ is a reasonable assumption.]