STEP 2010, Paper 1, Q2 – Solution (3 pages; 8/6/18)

$\frac{dy}{dx} = \frac{(x-b)-(x-a)}{(x-b)^2} e^x + \frac{x-a}{x-b} e^x$

$\frac{dy}{dx} = 0 \Rightarrow \frac{e^x}{(x-b)^2}\left(a - b + (x-a)(x-b)\right) = 0$

$\Rightarrow x^2 - (a+b)x + ab + a - b = 0 \quad (1)$

2 stationary points $\Rightarrow$ discriminant > 0

$\Rightarrow (a+b)^2 - 4(ab + a - b) > 0$

$\Rightarrow (a-b)^2 - 4(a-b) > 0$

$\Rightarrow (a-b)(a-b-4) > 0$

$\Rightarrow$ either $a - b - 4 > 0$ (when $a - b > 0$)

or $a - b < 0$ (when $a - b - 4 < 0$)

ie either $a - b < 0$ or $a - b > 4$, as required

(i) If $a = 0$ & $b = \frac{1}{2}$, (1) $\Rightarrow x^2 - \frac{1}{2}x - \frac{1}{2} = 0$

$\Rightarrow 2x^2 - x - 1 = 0 \Rightarrow (2x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{2}$ or 1

ie either side of the vertical asymptote of $x = \frac{1}{2}$

[The official sol'ns state: "we have $a - b = -\frac{1}{2} < 0$, so the curve has two stationary points by the first part of the question". Strictly speaking, in the first part of the question, we were asked to show (only) that the existence of two stationary points $\Rightarrow a - b < 0$ or $a - b > 4$; not that the existence of two stationary points $\Leftrightarrow$ $a - b < 0$ or $a - b > 4$ (although this is also true).]

To sketch $y = \frac{x}{(x-\frac{1}{2})} e^x$:

The curve crosses the axes at $(0, 0)$ only.

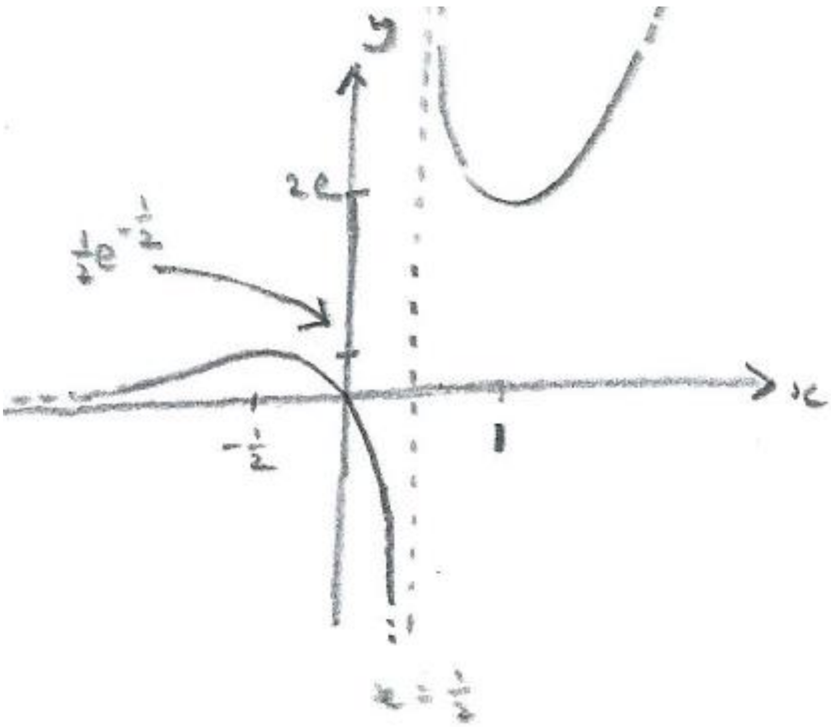
As $x \to \pm\infty,\ y \to e^x$

To see how the curve approaches the vertical asymptote:

$x = \frac{1}{2} + \delta \Rightarrow y > 0$ and $x = \frac{1}{2} - \delta \Rightarrow y < 0$

To establish the relative positions of the y-coordinates of the maximum and minimum:

$x = -\frac{1}{2} \Rightarrow y = \frac{1}{2}e^{-\frac{1}{2}}$ and $x = 1 \Rightarrow y = 2e > \frac{1}{2}e^{-\frac{1}{2}}$



(ii) If $a = \frac{9}{2}$ & $b = 0,\ y = \frac{x - \frac{9}{2}}{x} . e^x$

$(1) \Rightarrow x^2 - \frac{9x}{2} + \frac{9}{2} = 0 \Rightarrow 2x^2 - 9x + 9 = 0$

$\Rightarrow (2x - 3)(x - 3) = 0 \Rightarrow x = 3\ or\ \frac{3}{2}$

To see how the curve approaches the vertical asymptote:

$x = \delta \Rightarrow y < 0;\ x = -\delta \Rightarrow y > 0$

To establish the nature of the stationary points:

$x = \frac{3}{2} \Rightarrow y = -2e^{\frac{3}{2}}$; $x = 3 \Rightarrow y = -\frac{1}{2}e^3$

The question is: which of these is greater? [The official sol'ns don't seem to worry about this.]

$$\frac{2e^{\frac{3}{2}}}{\frac{1}{2}e^3} = 4e^{-\frac{3}{2}} = \frac{4}{e^{\frac{3}{2}}} = \left(\frac{16}{e^3}\right)^{\frac{1}{2}}$$

$$e^3 > (2 + 0.7)^3 > 8 + 3(2^2)(0.7) = 8 + 8.4 > 16$$

So $2e^{\frac{3}{2}} < \frac{1}{2}e^3$, and hence $(\frac{3}{2}, -2e^{\frac{3}{2}})$ is the maximum, and $(3, -\frac{1}{2}e^3)$ is the minimum.