STEP 2009, Paper 3, Q4 - Solution (2 pages; 7/6/18)

[Questions involving unfamiliar concepts (such as the Laplace transform) usually compensate by being reasonably straightforward (as is the case here).]

Let $L(f(t), s)$ denote the Laplace transform of f(t) with parameter s

(i) $L(e^{-bt}f(t), s) = \int_0^\infty e^{-st}e^{-bt}f(t)dt = \int_0^\infty e^{-(s+b)t}f(t)dt$

$= F(s+b)$ (since $s+b > 0$, as both s & b are > 0)

(ii) $L(f(at), s) = \int_0^\infty e^{-st}f(at)\,dt$

Let $u = at$, so that $du = adt$ and $dt = \frac{1}{a}du$, as $a \neq 0$

Then $L(f(at), s) = \int_0^\infty e^{-s\left(\frac{u}{a}\right)}f(u).\frac{1}{a}du = a^{-1}\int_0^\infty e^{-\left(\frac{s}{a}\right)u}f(u)du$

$= a^{-1}F\left(\frac{s}{a}\right)$ (since $s/a > 0$, as both s & a are > 0)

(iii) Integrating by Parts, $F(s) = \int_0^\infty e^{-st}f(t)dt =$
$\left[-\frac{1}{s}e^{-st}f(t)\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}f'(t)\,dt$

$= 0.f(\infty) + \frac{1}{s}f(0) + \frac{1}{s}L(f'(t), s)$

$\Rightarrow sF(s) = f(0) + L(f'(t), s)$ (provided $f(\infty)$ is finite)

$\Rightarrow L(f'(t), s) = sF(s) - f(0)$

(iv) $L(sint, s) = \int_0^\infty e^{-st}sint\,dt$

$= \left[-\frac{1}{s}e^{-st}sint\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}cost\,dt$

$= 0 + \frac{1}{s}\int_0^\infty e^{-st}cost\,dt$

$= \frac{1}{s}\left[-\frac{1}{s}e^{-st}cost\right]_0^\infty - \frac{1}{s}\int_0^\infty -\frac{1}{s}e^{-st}(-sint)dt$

$= \frac{1}{s^2} - \frac{1}{s^2}L(sint, s)$

Hence $s^2L(sint, s) = 1 - L(sint, s)$

$\Rightarrow L(sint, s)(s^2 + 1) = 1$

$\Rightarrow F(s) = L(sint, s) = \frac{1}{s^2+1}$

Let $f(t) = sint$ and $g(t) = e^{-pt}cosqt$

Then $g(t) = \frac{1}{q}e^{-pt}f'(qt)$

From (i), $L(g(t), s) = \frac{1}{q}L(f'(qt), s + p)$

Then, from (iii), $L(g(t), s) = \frac{1}{q}\{(s + p)L(f(qt), s + p) - f(q.0)\}$

$= \frac{1}{q}(s + p)L(f(qt), s + p)$

Then (ii)$\Rightarrow L(g(t), s) = \frac{1}{q}(s + p)q^{-1}L(f(t), \frac{s+p}{q})$

and (iv)$\Rightarrow L(g(t), s) = \frac{s+p}{q^2}.\frac{1}{\left(\frac{s+p}{q}\right)^2+1}$

Thus $F(s) = L(g(t), s) = \frac{s+p}{(s+p)^2+q^2}$