STEP 2009, Paper 3, Q11 - Solution (2 pages; 7/6/18)

(i) Conservation of momentum $\Rightarrow MV = M(1+bx)v$

$\Rightarrow V = (1+bx)\frac{dx}{dt} \Rightarrow \int V dt = \int (1+bx)dx$

$\Rightarrow Vt = x + \frac{1}{2}bx^2 + c$

$t = 0, x = 0 \Rightarrow c = 0$

$\Rightarrow bx^2 + 2x - 2Vt = 0$

$\Rightarrow x = \frac{-2+\sqrt{4-4b(-2Vt)}}{2b}$ (as $x > 0$)

Thus $x = \frac{\sqrt{1+2bVt}\ -1}{b}$

(ii) $Mf = \frac{d}{dt}(M(1+bx)v)$

$\Rightarrow \int f dt = (1+bx)v$

$\Rightarrow ft = (1+bx)v + c$

$t = 0, x = 0, v = V \Rightarrow 0 = V + c \Rightarrow c = -V$

Thus $ft + V = (1+bx)v$ and $v = \frac{ft+V}{1+bx}$, as required. (*)

So $\frac{dx}{dt} = \frac{ft+V}{1+bx}$ and hence $\int (1+bx)dx = \int ft + V\, dt$

$\Rightarrow x + \frac{1}{2}bx^2 = \frac{1}{2}ft^2 + Vt + c$

$x = 0, t = 0 \Rightarrow c = 0$

Thus $bx^2 + 2x - ft^2 - 2Vt = 0$

and $x = \frac{-2+\sqrt{4-4b(-ft^2-2Vt}}{b}$ (excluding $x < 0$ again)

So $x = \frac{\sqrt{1+bft^2+2bVt}\ -1}{b}$ (**)

[which agrees with the earlier result when $f = 0$]

Constant speed $\Rightarrow v = \frac{x}{t} = V$

Then $(*) \Rightarrow V = \frac{ft+V}{1+bVt} \Rightarrow V + bV^2t = ft + V$

$\Rightarrow bV^2 = f$; ie constant speed is possible, given this relation.

[see official sol'n for alternative method]

In the general case,

$(**) \Rightarrow v = \frac{dx}{dt} = \frac{1}{2b}(1 + bft^2 + 2bVt)^{-\frac{1}{2}}(2bft + 2bV)$

Consider $v^2 = \frac{(ft+V)^2}{1+bft^2+2bVt} = \frac{f^2t^2+2ftV+V^2}{1+bft^2+2bVt}$

$= \frac{f^2+\frac{2fV}{t}+\frac{V^2}{t^2}}{\frac{1}{t^2}+bf+2bV/t} \rightarrow \frac{f^2}{bf}$ as $t \rightarrow \infty$

[the limit of a quotient equals the quotient of the limits, provided that the limits are constants; ie not functions of t (in this case)]

Thus $v \rightarrow \sqrt{\frac{f}{b}}$ (a constant).

[see official sol'n for alternative method]