STEP 2009, Paper 2, Q11 – Solution (2 pages; 5/6/18)

Hint: To find T, note that t appears in $\frac{dv}{dt}$

Let F be the [variable] driving force of the engine.

Then $P = Fv$ and $N2L \Rightarrow F - (n+1)R = (n+1)Ma$

$\Rightarrow a = \frac{F}{(n+1)M} - \frac{R}{M} = \frac{P}{v(n+1)M} - \frac{R}{M}$ (1)

[If in doubt, use the previous part of the question; in this case to find T.]

From (1), $\frac{dv}{dt} = \frac{P-Rv(n+1)}{v(n+1)M}$,

so that $\int_0^V \frac{v(n+1)M}{P-Rv(n+1)} dv = \int_0^T dt = T$

$\Rightarrow T = \int_0^V \frac{\frac{M}{R}(Rv(n+1) - \frac{M}{R}P}{P-Rv(n+1)} + \frac{\frac{M}{R}P}{P-Rv(n+1)} dv$

$= \frac{M}{R}\int_0^V -1\, dv + \frac{PM}{R}\int_0^V \frac{1}{P-Rv(n+1)} dv$

$= -\frac{MV}{R} + \frac{PM}{R}\left[\frac{\ln(P-Rv(n+1))}{-R(n+1)}\right]_0^V$, provided $P > Rv(n+1)$

$= -\frac{MV}{R} + \frac{PM}{R^2(n+1)}\{lnP - \ln(P - RV(n+1)\}$

$= \frac{PM}{R^2(n+1)}\ln(\frac{P}{P-RV(n+1)}) - \frac{MV}{R}$

[There is a typo in the official sol'n, where v has been (incorrectly) written in place of V.]

As $P > Rv(n+1)$ is required throughout, a necessary condition is that $P > (n+1)RV$.

(i) $ln\left(\frac{P}{P-RV(n+1)}\right) = -ln\left(\frac{P-RV(n+1)}{P}\right) = -ln\left(1-\frac{RV(n+1)}{P}\right)$

$\approx -\left(-\frac{RV(n+1)}{P} - \frac{1}{2}\left(\frac{RV(n+1)}{P}\right)^2\right)$

Then $T \approx \frac{PM}{R^2(n+1)}\left(\frac{RV(n+1)}{P} + \frac{R^2V^2(n+1)^2}{2P^2}\right) - \frac{MV}{R}$

$$= \frac{MV}{R} + \frac{(n+1)V^2M}{2P} - \frac{MV}{R}$$

so that $PT \approx \frac{1}{2}(n+1)MV^2$

$\frac{(n+1)RV}{P} = \frac{(n+1)R}{F_V}$, where F_V is the driving force at speed V

So $\frac{(n+1)RV}{P}$ small $\Rightarrow$ total resistance, $(n+1)R$ is small compared to the final driving force, and hence to the (larger) driving force prior to that point.

If the resistance is small, then the KE gained, $\frac{1}{2}(n+1)MV^2$

$\approx$ work done by the driving force

$= \int F(x)dx = \int \frac{P}{\left(\frac{dx}{dt}\right)}dx = \int Pdt = PT$

(ii) Work done by driving force and resistance

$= PT - (n+1)RX = \frac{1}{2}(n+1)MV^2$

$\Rightarrow (n+1)RX = PT - \frac{1}{2}(n+1)MV^2$

$\Rightarrow X = \frac{PT - \frac{1}{2}(n+1)MV^2}{(n+1)R} = \frac{2PT-(n+1)MV^2}{2(n+1)R}$