STEP 2009, Paper 1, Q13 - Solution (2 pages; 11/4/21)

(i) Examples : GGGB...B,   B...BGGGB...B

$P(GGGB \ldots B) = \frac{3}{n+3} \cdot \frac{2}{n+2} \cdot \frac{1}{n+1}$

The different possibilities are all equally likely (to find

$P(B \ldots BGGGB \ldots B)$, where the Gs occur in positions

$r, r+1$ & $r+2$, we can first of all consider the probability of a girl being in the rth position etc).

So required probability $= \frac{3}{n+3} \cdot \frac{2}{n+2} \cdot \frac{1}{n+1} \cdot (n+1)$,

as there are $n+1$ possible positions for the 1st G (positions 1 to $n+1$)

ie $\frac{6}{(n+2)(n+3)}$

[Check: When $n = 0, \frac{6}{(n+2)(n+3)} = 1$; when $n = 1, \frac{6}{(n+2)(n+3)} = \frac{1}{2}$,

and $P(GGGB\ or\ BGGG) = \frac{1}{2}$, as these are half of the possible arrangements (the other two being $GBGG$ & $GGBG$).]

(ii) Let $P = GB$

Case 1: The last child is not a girl.

Examples: $BBPBBPBP$, $BBPPP$B

Case 2: The last child is a girl.

Examples: $BBPBBPBG$, $BBPBBPG$

The number of possibilities for Case 1 (with n boys & 3 girls, and therefore 3 Ps & $(n-3)$ Bs) is $\binom{n}{3}$

The number of possibilities for Case 2

(with 2 Ps , $(n-2)$Bs & the G at the end) is $\binom{n}{2}$

All the possibilities have probability $\frac{3}{n+3}\ .\frac{2}{n+2}\ .\frac{1}{n+1}$, as in (i).

So $P(K=1)=\frac{6}{(n+3)(n+2)(n+1)}\{\binom{n}{3}+\binom{n}{2}\}$

$=\frac{6}{(n+3)(n+2)(n+1)}\{\frac{n(n-1)(n-2)}{3!}+\frac{n(n-1)}{2!}\}$

$=\frac{n(n-1)(n-2+3)}{(n+3)(n+2)(n+1)}$

$=\frac{n(n-1))}{(n+3)(n+2)}$

(iii) $E(K)=P(K=1)+2P(K=2)+3P(K=3)$

and $P(K=2)=1-P(K=1)-P(K=3)$

So $E(K)=2-P(K=1)+P(K=3)$

$=2-\frac{n(n-1)}{(n+2)(n+3)}+\frac{6}{(n+2)(n+3)}$

$=\frac{2(n+2)(n+3)-n(n-1)+6}{(n+2)(n+3)}$

$=\frac{n^2+11n+18}{(n+2)(n+3)}=\frac{(n+2)(n+9)}{(n+2)(n+3)}=\frac{n+9}{n+3}$