STEP Solutions 2009

Mathematics

STEP 9465, 9470, 9475

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Step Mathematics (9465, 9470, 9475)

STEP I, Solutions
2009

Question 1

A proper factor *of an integer N is a positive integer, not 1 or N, that divides N.*

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors.

It is not by accident that this question writes "$3^2 \times 5^3$" and not "1125": it is aiming to suggest that it is much more straightforward to think about factors of a number if we are given its prime factorisation to begin with. Also, note that the question does not ask us to multiply out the factorisations at any point. In fact, there is no need to even give the factors explicitly if you do not need to.

Determining the proper factors of $3^2 \times 5^3$ is straightforward: any factor must be of the form $3^r \times 5^s$ with $0 \leqslant r \leqslant 2$ and $0 \leqslant s \leqslant 3$, giving the factors:

$$\begin{array}{ll}
3^0 \times 5^0 \ (= 1) & \text{(this is not a proper factor)} \\
3^0 \times 5^1 \ (= 5) & \\
3^0 \times 5^2 \ (= 25) & \\
3^0 \times 5^3 \ (= 125) & \\
3^1 \times 5^0 \ (= 3) & \\
3^1 \times 5^1 \ (= 15) & \\
3^1 \times 5^2 \ (= 75) & \\
3^1 \times 5^3 \ (= 375) & \\
3^2 \times 5^0 \ (= 9) & \\
3^2 \times 5^1 \ (= 45) & \\
3^2 \times 5^2 \ (= 225) & \\
3^2 \times 5^3 \ (= 1125) & \text{(this is not a proper factor)}
\end{array}$$

Therefore there are 10 proper factors in total.

Alternatively, we could simply note that there are 3 possible values for the power of 3 (namely 0, 1 and 2) and 4 for the power of 5 (namely 0, 1, 2 and 3), making $3 \times 4 = 12$ factors. Of these, two are not proper (1 and the number $3^2 \times 5^3$ itself), leaving $12 - 2 = 10$ proper factors.

(i) (cont.)

Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

Now that we have done this and understood how to count the factors of $3^2 \times 5^3$, we can answer the second part: the number of proper factors of $3^a \times 5^b$ is $(a+1)(b+1) - 2$, as the power of 3 in a factor can be 0, 1, ..., a, and the power of 5 can be 0, 1, ..., b. So we require $(a+1)(b+1) - 2 = 10$, or $(a+1)(b+1) = 12$. Here are the possibilities:

$a+1$	$b+1$	a	b	$n = 3^a \times 5^b$
1	12	0	11	$3^0 \times 5^{11}$
2	6	1	5	$3^1 \times 5^5$
3	4	2	3	$3^2 \times 5^3$
4	3	3	2	$3^3 \times 5^2$
6	2	5	1	$3^5 \times 5^1$
12	1	11	0	$3^{11} \times 5^0$

so there are 6 possibilities in total. This means that there are 5 other integers with the required properties.

We use these same ideas in part (ii).

> (ii) *Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.*

Following the same ideas as in part (i), let $n = 2^a \times 3^b \times 5^c \times 7^d \times \cdots$ be the prime factorisation of the positive integer n. (Note that we should use a letter other than N to distinguish our arbitrary integer from the special one that we seek.)

Then the number of factors of n is $(a+1)(b+1)(c+1)(d+1)\cdots$, and we must subtract 2 to get the number of proper factors. Assuming now that n has 426 proper factors, we must have

$$(a+1)(b+1)(c+1)(d+1)\cdots - 2 = 426,$$

so

$$(a+1)(b+1)(c+1)(d+1)\cdots = 428.$$

Now we can factorise $428 = 2^2 \times 107$, and 107 is prime. So the possible factorisations of 428 are $428 = 2 \times 214 = 4 \times 107 = 2 \times 2 \times 107$, so there can be at most three prime factors in n. We are seeking the smallest such n, so we choose the smallest possible primes, giving the smaller ones higher powers and larger ones smaller powers. So the smallest values of n for each possible factorisation of 428 are as follows:

With $428 = 428$:	$n = 2^{427}$
With $428 = 2 \times 214$:	$n = 2^{213} \times 3$
With $428 = 4 \times 107$:	$n = 2^{106} \times 3^3$
With $428 = 2 \times 2 \times 107$:	$n = 2^{106} \times 3 \times 5$

Since we seek the smallest possible value, our answer is clearly $2^{106} \times 3 \times 5$, as $2^{107} \times 3 > 3^3 = 27 > 3 \times 5 = 15$.

Question 2

> *A curve has the equation*
> $$y^3 = x^3 + a^3 + b^3\,,$$
> *where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is*
> $$b^2y - a^2x = a^3 + b^3\,.$$

Differentiating the equation of the curve with respect to x gives

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2.$$

Substituting $x = -a$ and $y = b$ gives $3b^2\,\frac{\mathrm{d}y}{\mathrm{d}x} = 3a^2$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = a^2/b^2$. Then the standard equation of a straight line gives

$$y - b = \frac{a^2}{b^2}(x + a),$$

which easily rearranges into the form $b^2y - a^2x = a^3 + b^3$, as required.

> *In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy*
> $$7x^3 - 3x^2 - 27x - 17 = 0\,.$$

In the case $a = 1$, $b = 2$, the curve has equation $y^3 = x^3 + 9$, and the tangent at $(-1, 2)$ has equation $4y - x = 9$. We therefore substitute $4y = x + 9$ into $y^3 = x^3 + 9$ as follows (after multiplying by 4^3):

$$\begin{aligned}
& & 64y^3 &= 64x^3 + 576 \\
&\Longrightarrow & (x+9)^3 &= 64x^3 + 576 \\
&\Longrightarrow & x^3 + 27x^2 + 243x + 729 &= 64x^3 + 576 \\
&\Longrightarrow & 63x^3 - 27x^2 - 243x - 153 &= 0 \\
&\Longrightarrow & 7x^3 - 3x^2 - 27x - 17 &= 0, \qquad \text{on dividing by 9,}
\end{aligned}$$

and this is the equation required.

> *Hence find positive integers p, q, r and s such that*
> $$p^3 = q^3 + r^3 + s^3\,.$$

Now our equation looks hard to solve, but we know that there is a solution at $x = -1$, as the curve and line are tangent at this point. In fact, since they are tangent, $x = -1$ must be a double root. So we can take out a factor of $(x + 1)^2$ to get

$$\begin{aligned}
& & (x+1)(7x^2 - 10x - 17) &= 0 \\
&\Longrightarrow & (x+1)^2(7x - 17) &= 0.
\end{aligned}$$

Thus either $x = -1$, which we aleady know, or $x = \frac{17}{7}$. Since this point lies on the line $4y - x = 9$, the y-coordinate is $\left(\frac{17}{7} + 9\right)/4 = \frac{20}{7}$. Thus, as this point also lies on the curve $y^3 = x^3 + a^3 + b^3$, we have
$$\left(\tfrac{20}{7}\right)^3 = \left(\tfrac{17}{7}\right)^3 + 1^3 + 2^3.$$
Now multiplying both sides by 7^3 gives us our required result:
$$20^3 = 17^3 + 7^3 + 14^3,$$
so a solution is $p = 20$, $q = 17$, $r = 7$, $s = 14$.

Question 3

> *(i) By considering the equation $x^2 + x - a = 0$, show that the equation $x = (a-x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0$.*

This looks somewhat confusing at first glance; why might $x = (a-x)^{\frac{1}{2}}$, which can be rearranged as the given quadratic, only have one solution whereas the quadratic can have two? But we must remember that this equation involves a square root, and by convention, this is the positive square root; therefore, real solutions must satisfy both $x \geqslant 0$ and $a-x \geqslant 0$, even if the quadratic has other solutions in addition.

Consider the equation

$$x = (a-x)^{\frac{1}{2}}. \tag{$*$}$$

If this is true, then squaring gives $x^2 = a - x$, or $x^2 + x - a = 0$. The solutions of this quadratic are given by

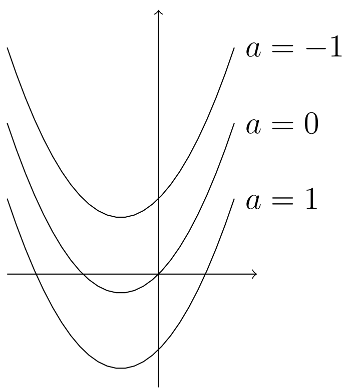
$$x = \frac{-1 \pm \sqrt{1+4a}}{2},$$

and these will be real solutions of $(*)$ if and only if $x \geqslant 0$ and $a - x \geqslant 0$, that is $0 \leqslant x \leqslant a$. But as $a - x = x^2$, we will always have $a - x \geqslant 0$ for real solutions of the quadratic, so we need only check that $x \geqslant 0$. For a solution to $(*)$, we therefore require the plus sign in the quadratic formula, and we also need $1 + 4a \geqslant 1$, so $a \geqslant 0$.

Thus, for $a < 0$, there are no real solutions to $(*)$, and for $a \geqslant 0$, $x = \left(-1+\sqrt{1+4a}\right)/2$ is the unique real solution.

An alternative approach is graphical. After we have shown that $(*)$ leads to the quadratic $x^2 + x - a = 0$, we see that the real solutions of $(*)$ correspond to those of the quadratic where $x \geqslant 0$. We can therefore sketch the graph of $y = x^2 + x - a$ and observe where the roots are.

The quadratic $y = x^2 + x$ has roots at $x = 0$ and $x = -1$ with a line of symmetry at $x = -\frac{1}{2}$. The equation $y = x^2 + x - a$ is a simple translation by a vertically downwards, like this:



It is therefore clear that, for $a \geqslant 0$, there is one root with $x \geqslant 0$, and for $a < 0$, there is no such root.

> *(i) (cont.)*
> *Find the number of distinct real solutions of the equation*
> $$x = \big((1+a)x - a\big)^{\frac{1}{3}}$$
> *in the cases that arise according to the value of a.*

Since cube-rooting is invertible, we have

$$x = \big((1+a)x - a\big)^{\frac{1}{3}} \iff x^3 = (1+a)x - a.$$

We are thus trying to solve the cubic equation $x^3 - (1+a)x + a = 0$. Inspection reveals one root, $x = 1$, so we can factorise the cubic as $(x-1)(x^2+x-a) = 0$. Using the discriminant of the quadratic factor, $1+4a$, we find that $x^2+x-a = 0$ has 0, 1 or 2 real roots according to whether $1+4a < 0$, $1+4a = 0$ or $1+4a > 0$, respectively.

Hence the original equation has 1 real root if $a < -\frac{1}{4}$, 2 distinct real roots if $a = -\frac{1}{4}$ (being $x = 1$ and $x = -\frac{1}{2}$), and 3 real roots if $a > -\frac{1}{4}$.

In the latter case, there is the possibility that they are not all distinct, though, if $x = 1$ is a root of $x^2 + x - a = 0$. This only happens when $a = 2$, and in this case, there are also only 2 *distinct* real roots.

> *(ii) Find the number of distinct real solutions of the equation*
> $$x = (b+x)^{\frac{1}{2}}$$
> *in the cases that arise according to the value of b.*

This is very similar to part (i), with the only difference being that this time we have $b + x$ rather than $a - x$. The argument should therefore be fairly similar to part (i).

Starting with the equation

$$x = (b+x)^{\frac{1}{2}}, \tag{†}$$

we again square this to get $x^2 = b + x$, or $x^2 - x - b = 0$.

From the first form, we see that to have any solutions, we must have $x \geqslant 0$ and $b + x \geqslant 0$. From the second, we see that the discriminant $1 + 4b \geqslant 0$ and $b + x = x^2 \geqslant 0$ as long as x is real. So if $b < -\frac{1}{4}$, there are no solutions.

The solutions to the quadratic are

$$x = \frac{1 \pm \sqrt{1+4b}}{2}.$$

In the case $b = -\frac{1}{4}$, the repeated solution is $x = \frac{1}{2} \geqslant 0$, so there is one solution in this case.

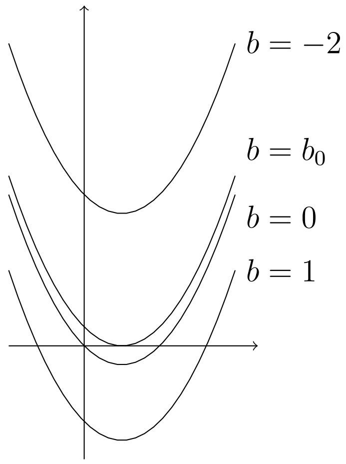
In the case $b > -\frac{1}{4}$, we still require $x \geqslant 0$ for solutions. The smaller root is

$$x = \frac{1 - \sqrt{1 + 4b}}{2},$$

which is negative if $b > 0$ and non-negative if $b \leqslant 0$.

Thus the equation (†) has no solutions if $b < -\frac{1}{4}$, one solution if $b = -\frac{1}{4}$ or $b > 0$, and two solutions if $-\frac{1}{4} < b \leqslant 0$.

An alternative approach to this part of the question is again to draw a graph of the functions involved. As in part (i), we draw the graph of $y = x^2 - x - b$ and determine how many roots it has with $x \geqslant 0$. The graph $y = x^2 - x$ has roots at $x = 0$ and $x = 1$, and the graph of $y = x^2 - x - b$ is a translation of this by $-b$ in the y-direction, as shown in this sketch:



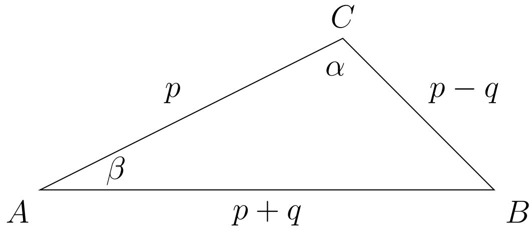
From these sketches it is clear that for $b < b_0$, there are no positive real solutions; at $b = b_0$ there is one (repeated) positive real solution, for $0 \geqslant b > b_0$ there are two non-negative real solutions, and for $b > 0$ there is one positive real solution. Finally, determining b_0 is easy: we want $x^2 - x - b_0$ to have a repeated root, and this means that $x^2 - x - b_0 = (x - \frac{1}{2})^2$, so that $b_0 = -\frac{1}{4}$.

Question 4

The sides of a triangle have lengths $p - q$, p and $p + q$, where $p > q > 0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1 - \cos\alpha)(1 - \cos\beta) = \cos\alpha + \cos\beta.$$

In situations like this, it's often useful to draw a sketch to gain some clarity about what is happening. Note that the largest angle is always opposite the longest side, and the smallest angle is always opposite the shortest side.



Using the cosine rule with the angles at A and C gives, respectively:

$$(p + q)^2 = p^2 + (p - q)^2 - 2p(p - q)\cos\alpha \quad (1)$$
$$(p - q)^2 = p^2 + (p + q)^2 - 2p(p + q)\cos\beta \quad (2)$$

Then we need to manipulate these two equations in order to reach the desired result. There are several ways to do this; we show two of them.

Approach 1: Determining the cosines and substituting

Equation (1) gives, upon rearranging:

$$\begin{aligned}\cos\alpha &= \frac{p^2 + (p - q)^2 - (p + q)^2}{2p(p - q)} \\ &= \frac{p^2 + (p^2 - 2pq + q^2) - (p^2 + 2pq + q^2)}{2p(p - q)} \\ &= \frac{p^2 - 4pq}{2p(p - q)} \\ &= \frac{p - 4q}{2(p - q)}. \quad (3)\end{aligned}$$

Likewise, equation (2) yields:

$$\begin{aligned}
\cos\beta &= \frac{p^2+(p+q)^2-(p-q)^2}{2p(p+q)} \\
&= \frac{p^2+(p^2+2pq+q^2)-(p^2-2pq+q^2)}{2p(p+q)} \\
&= \frac{p^2+4pq}{2p(p+q)} \\
&= \frac{p+4q}{2(p+q)}.
\end{aligned} \tag{4}$$

Using equations (3) and (4), we now evaluate $4(1-\cos\alpha)(1-\cos\beta)$ and $\cos\alpha+\cos\beta$:

$$\begin{aligned}
4(1-\cos\alpha)(1-\cos\beta) &= 4\left(1-\frac{p-4q}{2(p-q)}\right)\left(1-\frac{p+4q}{2(p+q)}\right) \\
&= 4.\frac{p+2q}{2(p-q)}.\frac{p-2q}{2(p+q)} \\
&= \frac{p^2-4q^2}{p^2-q^2}
\end{aligned}$$

and

$$\begin{aligned}
\cos\alpha+\cos\beta &= \frac{p-4q}{2(p-q)}+\frac{p+4q}{2(p+q)} \\
&= \frac{(p-4q)(p+q)+(p+4q)(p-q)}{2(p-q)(p+q)} \\
&= \frac{p^2-3pq-4q^2+p^2+3pq-4q^2}{2(p^2-q^2)} \\
&= \frac{2(p^2-4q^2)}{2(p^2-q^2)} \\
&= \frac{p^2-4q^2}{p^2-q^2}.
\end{aligned}$$

Therefore we have the required equality

$$4(1-\cos\alpha)(1-\cos\beta) = \cos\alpha+\cos\beta. \tag{$*$}$$

Approach 2: Determining q/p and equating

From equation (1) above, we can expand to get

$$p^2+2pq+q^2 = p^2+p^2-2pq+q^2-2p(p-q)\cos\alpha,$$

so that

$$p^2-4pq = 2p(p-q)\cos\alpha.$$

We now divide by p^2 to get

$$1-4q/p = 2(1-q/p)\cos\alpha.$$

We can now rearrange this to find q/p:

$$(2\cos\alpha - 4)q/p = 2\cos\alpha - 1,$$

so

$$\frac{q}{p} = \frac{2\cos\alpha - 1}{2\cos\alpha - 4}.$$

Doing the same for equation (2) gives:

$$p^2 - 2pq + q^2 = p^2 + p^2 + 2pq + q^2 - 2p(p+q)\cos\beta,$$

so that

$$p^2 + 4pq = 2p(p+q)\cos\beta.$$

Again, dividing by p^2 brings us to

$$1 + 4q/p = 2(1 + q/p)\cos\beta,$$

yielding

$$(4 - 2\cos\beta)q/p = 2\cos\beta - 1,$$

so

$$\frac{q}{p} = \frac{2\cos\beta - 1}{4 - 2\cos\beta}.$$

Equating these two expressions for q/p now gives us

$$\frac{2\cos\alpha - 1}{2\cos\alpha - 4} = \frac{2\cos\beta - 1}{4 - 2\cos\beta},$$

so that (cross-multiplying and dividing by two):

$$(2\cos\alpha - 1)(2 - \cos\beta) = (\cos\alpha - 2)(2\cos\beta - 1).$$

Now we expand the brackets to get

$$4\cos\alpha - 2 - 2\cos\alpha\cos\beta + \cos\beta = 2\cos\alpha\cos\beta - 4\cos\beta - \cos\alpha + 2,$$

so that

$$4 - 4\cos\alpha - 4\cos\beta + 4\cos\alpha\cos\beta = \cos\alpha + \cos\beta,$$

and the left hand side factorises to give us our desired result:

$$4(1 - \cos\alpha)(1 - \cos\beta) = \cos\alpha + \cos\beta.$$

In the case $\alpha = 2\beta$, show that $\cos\beta = \frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

Substituting $\alpha = 2\beta$ into $(*)$ gives:

$$4(1 - \cos 2\beta)(1 - \cos\beta) = \cos 2\beta + \cos\beta.$$

We use the double angle formula for $\cos 2\beta$ to write this expression in terms of $\cos\beta$, giving:

$$4(2 - 2\cos^2\beta)(1 - \cos\beta) = 2\cos^2\beta + \cos\beta - 1,$$

so

$$8(1 + \cos\beta)(1 - \cos\beta)^2 = (2\cos\beta - 1)(\cos\beta + 1).$$

Since $\cos\beta \neq -1$, we can divide by $\cos\beta + 1$ to get

$$8(1 - \cos\beta)^2 = 2\cos\beta - 1,$$

so we can rearrange to get

$$8\cos^2\beta - 18\cos\beta + 9 = 0,$$

which factorises as

$$(4\cos\beta - 3)(2\cos\beta - 3) = 0.$$

Since $\cos\beta = \frac{3}{2}$ is impossible, we must have $\cos\beta = \frac{3}{4}$, as required.

We now substitute this result into equation (2) to get

$$(p - q)^2 = p^2 + (p + q)^2 - 2p(p + q).\tfrac{3}{4}.$$

Expanding this gives

$$p^2 - 2pq + q^2 = p^2 + p^2 + 2pq + q^2 - \tfrac{3}{2}p^2 - \tfrac{3}{2}pq,$$

so

$$\tfrac{1}{2}p^2 - \tfrac{5}{2}pq = 0,$$

which gives $p = 5q$. Hence the side lengths are $p - q = 4q$, $p = 5q$ and $p + q = 6q$, which are in the ratio $4 : 5 : 6$.

An alternative way to do this last part is as follows. We have $\alpha = 2\beta$, so $\cos\alpha = 2\cos^2\beta - 1 = 2 \cdot (\frac{3}{4})^2 - 1 = \frac{1}{8}$. It follows that $\sin\alpha = \frac{1}{8}\sqrt{63} = \frac{3}{8}\sqrt{7}$ and $\sin\beta = \frac{1}{4}\sqrt{7}$. We can now use the sine rule to get

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

or $a/c = \sin A/\sin C$. It follows that

$$\begin{aligned}\frac{p - q}{p + q} &= \frac{\sin\beta}{\sin\alpha} \\ &= \frac{\frac{1}{4}\sqrt{7}}{\frac{3}{8}\sqrt{7}} \\ &= \frac{2}{3},\end{aligned}$$

giving $3(p - q) = 2(p + q)$, or $p = 5q$. The rest of the result follows as above.

A third way of doing this, and arguably the simplest, is to substitute into equation (4), which gives:

$$\frac{3}{4} = \frac{p + 4q}{2(p + q)}.$$

Multiplying both sides by $4(p + q)$ to clear the fractions gives

$$3(p + q) = 2(p + 4q),$$

so that $p = 5q$. The rest of the argument again follows as above.

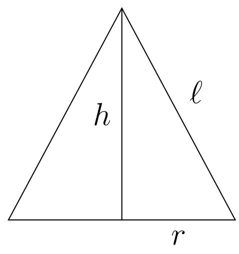
Question 5

A right circular cone has base radius r, height h and slant length ℓ. Its volume V, and the area A of its curved surface, are given by

$$V = \tfrac{1}{3}\pi r^2 h, \qquad A = \pi r \ell.$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $\ell = \sqrt{3}\,r$.

Since A is fixed and r is allowed to vary, we rearrange $A = \pi r \ell$ as $\ell = A/\pi r$. Also, we can draw a side view of the cone to determine the relationship between r, h and ℓ:



So clearly, $\ell^2 = h^2 + r^2$.

Substituting this into $\ell = A/\pi r$ gives

$$\ell^2 = h^2 + r^2 = \frac{A^2}{\pi^2 r^2},$$

which we can rearrange to give h^2 in terms of r and A:

$$h^2 = \frac{A^2}{\pi^2 r^2} - r^2.$$

Now $V = \frac{1}{3}\pi r^2 h$, so we have

$$\begin{aligned} V^2 &= \tfrac{1}{9}\pi^2 r^4 h^2 \\ &= \frac{\pi^2 r^4}{9}\left(\frac{A^2}{\pi^2 r^2} - r^2\right) \\ &= \tfrac{1}{9}(A^2 r^2 - \pi^2 r^6). \end{aligned}$$

(Working with V^2 rather than just V allows us to avoid square roots.)

Differentiating with respect to r gives

$$2V\frac{\mathrm{d}V}{\mathrm{d}r} = \tfrac{1}{9}(2A^2 r - 6\pi^2 r^5).$$

When V is at its stationary value, $\mathrm{d}V/\mathrm{d}r = 0$, so we require $2A^2 r - 6\pi^2 r^5 = 0$. As $r \neq 0$, we must have $6\pi^2 r^4 = 2A^2$, or $A^2 = 3\pi^2 r^4$, as wanted.

Substituting this into our formula for ℓ^2 gives

$$\begin{aligned}\ell^2 &= \frac{A^2}{\pi^2 r^2}\\ &= \frac{3\pi^2 r^4}{\pi^2 r^2}\\ &= 3r^2,\end{aligned}$$

so $\ell = \sqrt{3}\, r$, as we wanted to show.

> (ii) *Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.*

We have $\ell^2 = h^2 + r^2 = A^2/\pi^2 r^2$ and $V = \frac{1}{3}\pi r^2 h$. This time, V is fixed, so $h = 3V/\pi r^2$. Thus

$$\begin{aligned}A^2 &= \pi^2 r^2 (h^2 + r^2)\\ &= \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)\\ &= \frac{9V^2}{r^2} + \pi^2 r^4\end{aligned}$$

Differentiating as before gives

$$2A\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{18V^2}{r^3} + 4\pi^2 r^3,$$

so $\mathrm{d}A/\mathrm{d}r = 0$ when $4\pi^2 r^6 = 18V^2$, so $2\pi r^3 = 3V\sqrt{2}$. Finally, substituting this into our formula $h = 3V/\pi r^2$ gives

$$\begin{aligned}h &= \frac{2\pi r^3/\sqrt{2}}{\pi r^2}\\ &= \sqrt{2}\, r.\end{aligned}$$

Question 6

> *(i) Show that, for $m > 0$,*
>
> $$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m\,.$$

We note that the numerator of the fraction (x^2) has a higher degree than the denominator ($x+1$), so we divide them first, getting $x^2 = (x+1)(x-1)+1$, so our integral becomes

$$\begin{aligned}
\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x &= \int_{1/m}^{m} x - 1 + \frac{1}{x+1}\,\mathrm{d}x \\
&= \Big[\tfrac{1}{2}x^2 - x + \ln|x+1|\Big]_{1/m}^{m} \\
&= \big(\tfrac{1}{2}m^2 - m + \ln|m+1|\big) - \big(\tfrac{1}{2}m^{-2} - m^{-1} + \ln|(1/m)+1|\big) \\
&= \frac{m^4 - 2m^3 - 1 + 2m}{2m^2} + \ln\frac{m+1}{(1/m)+1} \\
&= \frac{(m+1)(m^3 - 3m^2 + 3m - 1)}{2m^2} + \ln\frac{m(m+1)}{1+m} \\
&= \frac{(m+1)(m-1)^3}{2m^2} + \ln m.
\end{aligned}$$

(An alternative approach is to use the substitution $u = x+1$, which leads to exactly the same result.)

> *(ii) Show by means of a substitution that*
>
> $$\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

Comparing the two integrals suggests that we should try the substitution $u = 1/x$. If we do this, we get $x = 1/u$ and $\mathrm{d}x/\mathrm{d}u = -1/u^2$. Also, the limits $x = 1/m$ and $x = m$ become $u = m$ and $u = 1/m$ respectively. So we have

$$\begin{aligned}
\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x &= \int_{m}^{1/m} \frac{1}{(1/u)^n(1/u+1)}\,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \int_{m}^{1/m} \frac{u^n}{1/u+1}\,\frac{-1}{u^2}\,\mathrm{d}u \\
&= -\int_{m}^{1/m} \frac{u^n}{u(1+u)}\,\mathrm{d}u \\
&= \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u.
\end{aligned}$$

(iii) Evaluate:

(a) $$\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x\,.$$

This clearly relies on the earlier parts of the question, where $m = 2$. We can break the integral into two parts and use the results of (i) and (ii) as follows:

$$\begin{aligned}
\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x &= \int_{1/2}^{2} \frac{x^5}{x^3(x+1)}\,\mathrm{d}x + 3\int_{1/2}^{2} \frac{1}{x^3(x+1)}\,\mathrm{d}x \\
&= \int_{1/2}^{2} \frac{x^2}{x+1}\,\mathrm{d}x + 3\int_{1/2}^{2} \frac{u^2}{u+1}\,\mathrm{d}u \\
&= 4\int_{1/2}^{2} \frac{x^2}{x+1}\,\mathrm{d}x \\
&= 4\left(\frac{(m+1)(m-1)^3}{2m^2} + \ln m\right) \\
&= 4(\tfrac{3}{8} + \ln 2) \\
&= \tfrac{3}{2} + 4\ln 2.
\end{aligned}$$

An alternative way to approach this question is to ignore what has gone before and to use partial fractions. We first divide to get

$$\frac{x^5+3}{x^3(x+1)} = x - 1 + \frac{x^3+3}{x^3(x+1)}$$

and then express the final term using partial fractions:

$$\frac{x^3+3}{x^3(x+1)} = \frac{3}{x} - \frac{3}{x^2} + \frac{3}{x^3} - \frac{2}{x+1}.$$

Integrating then gives:

$$\begin{aligned}
\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x &= \int_{1/2}^{2} x - 1 + \frac{3}{x} - \frac{3}{x^2} + \frac{3}{x^3} - \frac{2}{x+1}\,\mathrm{d}x \\
&= \left[\frac{x^2}{2} - x + 3\ln x + \frac{3}{x} - \frac{3}{2x^2} - 2\ln(x+1)\right]_{1/2}^{2} \\
&= (2 - 2 + 3\ln 2 + \tfrac{3}{2} - \tfrac{3}{8} - 2\ln 3) - (\tfrac{1}{8} - \tfrac{1}{2} + 3\ln\tfrac{1}{2} + 6 - 6 - 2\ln\tfrac{3}{2}) \\
&= \tfrac{3}{2} + 4\ln 2.
\end{aligned}$$

(iii) Evaluate:

(b) $$\int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x\,.$$

It is no longer so obvious how to proceed, as the limits are not of the form $1/m$ to m. So we break up the integral as in (a) and repeat the substitution of part (ii) once again, noting that the only change here is that the limits are different. We have:

$$\int_1^2 \frac{x^5}{x^3(x+1)}\,\mathrm{d}x = \int_1^2 \frac{x^2}{x+1}\,\mathrm{d}x,$$

and there is not much we can do at this point, short of evaluating this integral as in part (i).

Next, we have

$$\begin{aligned}\int_1^2 \frac{x^3}{x^3(x+1)}\,\mathrm{d}x &= \int_1^2 \frac{1}{x+1}\,\mathrm{d}x\\ &= \big[\ln|x+1|\big]_1^2\\ &= \ln 3 - \ln 2.\end{aligned}$$

The third part gives us

$$\int_1^2 \frac{1}{x^3(x+1)}\,\mathrm{d}x = \int_{1/2}^1 \frac{u^2}{u+1}\,\mathrm{d}u,$$

by using the substitution of part (ii), but noting the the limits transform into $1/1 = 1$ and $1/2$, which are then reversed by the minus sign.

Adding all three terms and using part (i) with $m = 2$ now gives:

$$\begin{aligned}\int_1^2 \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x &= \int_1^2 \frac{x^2}{x+1}\,\mathrm{d}x + \ln 3 - \ln 2 + \int_{1/2}^1 \frac{u^2}{u+1}\,\mathrm{d}u\\ &= \int_{1/2}^2 \frac{x^2}{x+1}\,\mathrm{d}x + \ln 3 - \ln 2\\ &= \tfrac{3}{8} + \ln 2 + \ln 3 - \ln 2\\ &= \tfrac{3}{8} + \ln 3.\end{aligned}$$

Again, this question can also be approached using partial fractions. Dividing gives

$$\frac{x^5+x^3+1}{x^3(x+1)} = x - 1 + \frac{2x^3+1}{x^3(x+1)}$$

and then partial fractions expansion gives us:

$$\frac{2x^3+1}{x^3(x+1)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x+1}.$$

We now integrate to reach our final answer:

$$\begin{aligned}\int_1^2 \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x &= \int_1^2 x - 1 + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x+1}\,\mathrm{d}x\\ &= \left[\frac{x^2}{2} - x + \ln x + \frac{1}{x} - \frac{1}{2x^2} + \ln(x+1)\right]_1^2\\ &= \left(2 - 2 + \ln 2 + \tfrac{1}{2} - \tfrac{1}{8} + \ln 3\right) - \left(\tfrac{1}{2} - 1 + 0 + 1 - \tfrac{1}{2} + \ln 2\right)\\ &= \tfrac{3}{8} + \ln 3.\end{aligned}$$

Question 7

Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx \,\mathrm{d}x = \frac{1}{m^2+1}\left(\mathrm{e}^{2\pi}-1\right).$$

This is a standard integral with standard techniques for solving it. The approach used here is not the only possible one, but is fairly general.

We write $I = \int_0^{2\pi} \mathrm{e}^x \cos mx \,\mathrm{d}x$ and apply integration by parts twice, taking great care of the signs, as follows:

$$\begin{aligned}
I &= \int_0^{2\pi} \mathrm{e}^x \cos mx \,\mathrm{d}x \\
&= \left[\mathrm{e}^x \cdot \frac{1}{m}\sin mx\right]_0^{2\pi} - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m}\sin mx \,\mathrm{d}x \\
&= \left(\tfrac{1}{m}\mathrm{e}^{2\pi}\sin 2\pi m\right) - \left(\tfrac{1}{m}\mathrm{e}^0 \sin 0\right) - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m}\sin mx \,\mathrm{d}x \\
&= 0 - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m}\sin mx \,\mathrm{d}x \\
&= -\left[\mathrm{e}^x \cdot \frac{1}{m^2}(-\cos mx)\right]_0^{2\pi} + \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m^2}(-\cos mx)\,\mathrm{d}x \\
&= \left(\frac{1}{m^2}\mathrm{e}^{2\pi}\cos 2\pi m\right) - \left(\frac{1}{m^2}\mathrm{e}^0\cos 0\right) - \int_0^{2\pi} \mathrm{e}^x \cdot \frac{1}{m^2}\cos mx \,\mathrm{d}x \\
&= \frac{1}{m^2}\left(\mathrm{e}^{2\pi}-1\right) - \frac{1}{m^2}I.
\end{aligned}$$

Multiplying throughout by m^2 gives

$$m^2 I = \left(\mathrm{e}^{2\pi}-1\right) - I,$$

and now adding I to both sides and dividing by m^2+1 gives the desired result.

We performed these integrations by writing $\int \mathrm{e}^x \cos mx \,\mathrm{d}x$ in the form $\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$, where $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos mx$ and $v = \mathrm{e}^x$. We could equally well have chosen $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ and $v = \cos mx$, and would have ended up with the same conclusion.

(i) Expand $\cos(A+B)+\cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x \,\mathrm{d}x = \tfrac{19}{650}\left(\mathrm{e}^{2\pi}-1\right).$$

We have

$$\begin{aligned}\cos(A+B)+\cos(A-B) &= \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B\\ &= 2\cos A\cos B.\end{aligned}$$

Matching this to the integral we have been given, we set $A = 6x$ and $B = x$ (so that $A - B$ is positive, though this is not critical), giving

$$2\cos x\cos 6x = \cos 7x + \cos 5x.$$

Thus our integral becomes

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^x \cos x\cos 6x \;\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x(\cos 7x + \cos 5x)\;\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{1}{7^2+1}(\mathrm{e}^{2\pi}-1) + \frac{1}{5^2+1}(\mathrm{e}^{2\pi}-1)\right)\\ &= \frac{1}{2}\left(\frac{1}{50}+\frac{1}{26}\right)(\mathrm{e}^{2\pi}-1)\\ &= \frac{1}{2}\cdot\frac{26+50}{1300}(\mathrm{e}^{2\pi}-1)\\ &= \tfrac{19}{650}(\mathrm{e}^{2\pi}-1),\end{aligned}$$

as required.

(ii) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x\sin 4x\cos x \;\mathrm{d}x$.

We are clearly asked to do the same type of trick again. Here is one way to proceed.

We are looking for the product of two sines, and this appears in the compound angle formula for cosine. So we consider

$$\begin{aligned}\cos(A+B)-\cos(A-B) &= \cos A\cos B - \sin A\sin B - \cos A\cos B - \sin A\sin B\\ &= -2\sin A\sin B.\end{aligned}$$

We can therefore write $2\sin 2x\sin 4x = \cos 2x - \cos 6x$, using $A = 4x$ and $B = 2x$. This gives the integral as

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^x \sin 2x\sin 4x\cos x \;\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x(\cos 2x - \cos 6x)\cos x\;\mathrm{d}x\\ &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x\cos 2x\cos x\;\mathrm{d}x - \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x\cos 6x\cos x\;\mathrm{d}x.\end{aligned}$$

Now the second integral is exactly the one we evaluated in (i), and the first integral can be approached in the same way, with $A = 2x$ and $B = x$, so $2\cos 2x\cos x = \cos 3x + \cos x$.

Therefore

$$\begin{aligned}
\int_0^{2\pi} \mathrm{e}^x \cos 2x \cos x \ \mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x(\cos 3x + \cos x) \ \mathrm{d}x \\
&= \frac{1}{2}\left(\frac{1}{3^2+1}\big(\mathrm{e}^{2\pi}-1\big) + \frac{1}{1^2+1}\big(\mathrm{e}^{2\pi}-1\big)\right) \\
&= \frac{1}{2}\left(\frac{1}{10}+\frac{1}{2}\right)\big(\mathrm{e}^{2\pi}-1\big) \\
&= \tfrac{3}{10}\big(\mathrm{e}^{2\pi}-1\big).
\end{aligned}$$

Finally, subtracting the two integrals gives

$$\begin{aligned}
\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x \ \mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x \cos 2x \cos x \ \mathrm{d}x - \frac{1}{2}\int_0^{2\pi} \mathrm{e}^x \cos 6x \cos x \ \mathrm{d}x \\
&= \tfrac{1}{2}\cdot\tfrac{3}{10}\big(\mathrm{e}^{2\pi}-1\big) - \tfrac{1}{2}\cdot\tfrac{19}{650}\big(\mathrm{e}^{2\pi}-1\big) \\
&= \tfrac{1}{2}\big(\tfrac{195}{650}-\tfrac{19}{650}\big)\big(\mathrm{e}^{2\pi}-1\big) \\
&= \tfrac{1}{2}\cdot\tfrac{176}{650}\big(\mathrm{e}^{2\pi}-1\big) \\
&= \tfrac{44}{325}\big(\mathrm{e}^{2\pi}-1\big).
\end{aligned}$$

Question 8

(i) *The equation of the circle C is*

$$(x-2t)^2 + (y-t)^2 = t^2,$$

where t is a positive number. Show that C touches the line $y = 0$.

This is a circle with centre $(2t, t)$ and radius t, therefore it touches the x-axis, which is distance t from the centre.

Alternatively, we are looking to solve the simultaneous equations

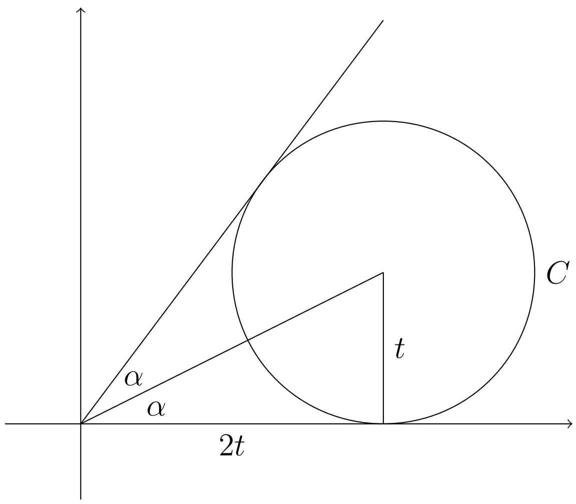
$$\begin{aligned}(x-2t)^2 + (y-t)^2 &= t^2 \\ y &= 0.\end{aligned}$$

Substituting the second equation into the first gives $(x-2t)^2 + t^2 = t^2$, or $(x-2t)^2 = 0$. Since this only has one solution, $x = 2t$, the line must be tangent to the circle.

(i) *(cont.)*

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y = 4x$.

We begin by drawing a sketch of the situation.



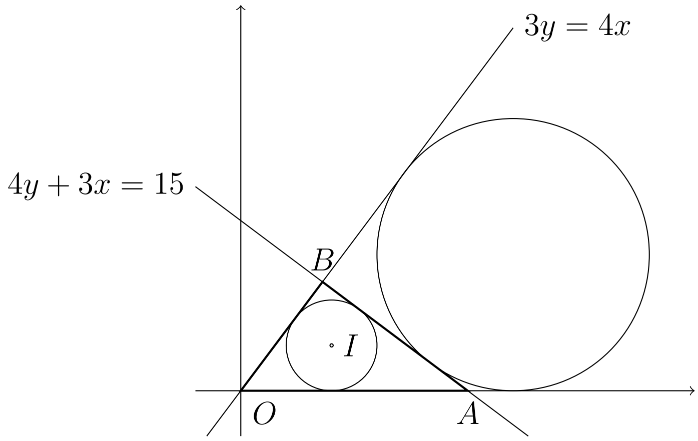
Clearly, therefore, $\tan\alpha = t/2t = \frac{1}{2}$, so

$$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha} \\ &= \frac{1}{1-\frac{1}{4}} \\ &= \tfrac{4}{3}.\end{aligned}$$

From the sketch, it is clear that by symmetry, the line through the origin which makes an angle of 2α with the x-axis touches the circle C. Since the gradient of this line is $\tan 2\alpha = \frac{4}{3}$ and it passes through the origin, it has equation $y = \frac{4}{3}x$, or $3y = 4x$.

> (ii) *Find the equation of the incircle of the triangle formed by the lines $y = 0$, $3y = 4x$ and $4y + 3x = 15$.*
>
> ***Note:*** *The incircle of a triangle is the circle, lying totally inside the triangle, that touches all three sides.*

This circle has the properties described in part (i), in that it touches both the x-axis (i.e., the line $y = 0$) and the line $3y = 4x$, and it lies above the x-axis, so it must have the form given. The only remaining condition is that it must touch the line $4y+3x = 15$. Two circles with centre $(2t, t)$ touch these three lines, but only one of them lies within the triangle, as illustrated on this sketch, so we must take the one with the smaller value of t:



Approach 1: Algebraic substitution

Substituting $4y + 3x = 15$ into the equation of C will give us the intersections of the line and the circle. The line will be a tangent to C if and only if the discriminant of the resulting quadratic equation is 0. Doing this, from the equation of C:

$$(x - 2t)^2 + (y - t)^2 = t^2$$

we get:

$$(x - 2t)^2 + ((15 - 3x)/4 - t)^2 = t^2.$$

Multiplying both sides by 4^2 and expanding gives:

$$16x^2 - 64tx + 64t^2 + (15 - 3x)^2 - 8(15 - 3x)t + 16t^2 = 16t^2,$$

so

$$16x^2 - 64tx + 64t^2 + 225 - 90x + 9x^2 - 120t + 24tx + 16t^2 = 16t^2.$$

Collecting terms in x gives:

$$25x^2 - (90 + 40t)x + 225 - 120t + 64t^2 = 0.$$

Since this has a repeated root, as the line is required to be tangent to the circle, the discriminant must be zero, so

$$(90 + 40t)^2 - 4 \times 25(225 - 120t + 64t^2) = 0,$$

so

$$(9 + 4t)^2 - (225 - 120t + 64t^2) = 0,$$

or

$$16t^2 + 72t + 81 - (225 - 120t + 64t^2) = 0,$$

giving

$$-48t^2 + 192t - 144 = 0.$$

Dividing all the terms by -48 gives $t^2 - 4t + 3 = 0$, which factorises as $(t - 1)(t - 3) = 0$. Thus the two circles in the sketch are given by $t = 1$ and $t = 3$, and the incircle is clearly the one with $t = 1$. So the incircle has equation

$$(x - 2)^2 + (y - 1)^2 = 1.$$

Alternative 2: Distance of a point from a line

In the formula book, we are given the result:

The perpendicular distance of (α, β, γ) *from* $n_1x+n_2y+n_3z+d = 0$ *is* $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$.

Therefore the distance of the centre of the circle, $(2t, t, 0)$, from the line $3x+4y+0z-15 = 0$ is given by

$$\frac{|3 \times 2t + 4 \times t + 0 - 15|}{\sqrt{3^2 + 4^2 + 0^2}} = \frac{|10t - 15|}{5} = |2t - 3|.$$

But the line is tangent to the circle, which has radius t, so we must have

$$|2t - 3| = t.$$

This equation has two solutions: $2t - 3 = t$ gives $t = 3$, and $2t - 3 = -t$ gives $t = 1$. As we require the smaller solution, we must have $t = 1$, so that the incircle has equation

$$(x - 2)^2 + (y - 1)^2 = 1.$$

Alternative 3: Finding another angle bisector

As in part (i), using the notation in the diagram above, the line from A to I bisects the angle OAB. So if we write $O\hat{A}I = \beta$, then $O\hat{A}B = 2\beta$.

Now $\tan 2\beta = \frac{3}{4}$, so we solve

$$\tan 2\beta = \frac{2\tan\beta}{1 - \tan^2\beta} = \frac{3}{4}$$

to find $\tan\beta$. We get the quadratic equation

$$3 - 3\tan^2\beta = 8\tan\beta,$$

which factorises as $(3\tan\beta - 1)(\tan\beta + 3) = 0$, so $\tan\beta = \frac{1}{3}$ or $\tan\beta = -3$. But β is acute, so $\tan\beta = \frac{1}{3}$. Thus the line AI has equation

$$y - 0 = -\tfrac{1}{3}(x - 5)$$

which can be written as $3y + x = 5$.

The point I has coordinates $(2t, t)$, and substituting this in gives $3t + 2t = 5$, so $t = 1$ as before.

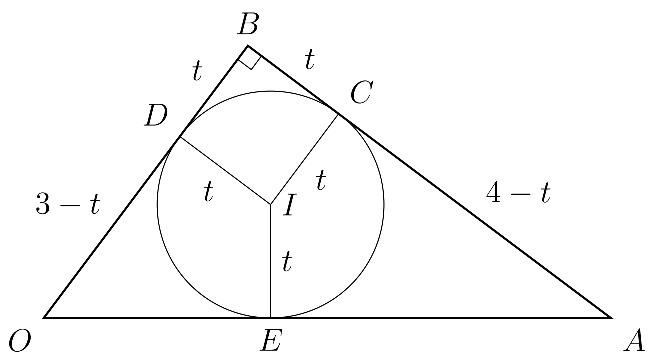
Alternative 4: Using Euclidean geometry

We find that the intersection of the line $4y + 3x = 15$ with the x-axis is at $(5, 0)$. We can also find the intersection of the two lines $4y + 3x = 15$ and $3y = 4x$ by solving them simultaneously:

$$\begin{aligned} 12y + 9x &= 45 \\ 12y - 16x &= 0, \end{aligned}$$

so $25x = 45$, or $x = \frac{9}{5}$, so $y = \frac{12}{5}$. Thus the side of the triangle from the origin to the point of intersection has length 3, the side from $(5, 0)$ to the point of intersection has length 4 and the base has length 5 (using Pythagoras), so we have a 3-4-5 triangle. (The triangle is right-angled as $4y + 3x = 15$ and $3y = 4x$ are perpendicular.)

Consider now this figure, where we have drawn radii from the incentre to the three sides of the triangle. Note that, since the triangle's sides are tangents to the circle, the radii meet at right angles, so $ICBD$ is a square with all sides equal to the radius, t.



Now $AC = AE = 4 - t$ since these are both tangents from A to the circle; likewise, $OE = OD = 3 - t$, so $OA = OE + EA = (3 - t) + (4 - t) = 7 - 2t$. But $OA = 5$, so $2t = 2$ and $t = 1$, giving the radius and hence the equation of the incircle.

Alternative 5: More Euclidean geometry

We begin in the same way by finding the side lengths of the triangle. We can then quote the result that if r is the incircle radius and the side lengths of the triangle are a, b, c, then these are related by

$$\text{Area} = \tfrac{1}{2}r(a + b + c).$$

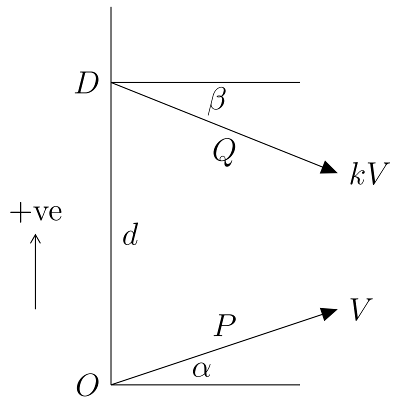
In this case, the area is $\frac{1}{2} \times 3 \times 4 = 6$ and $\frac{1}{2}(a + b + c) = 6$, so $r = 1$ as required.

Question 9

Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection above the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection below the horizontal is β. The particles collide at time T after projection.
Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation

$$(k^2 - 1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0.$$

We begin by drawing a sketch showing the initial situation.



For components, we will write u_P and u_Q for the horizontal components of the velocities of P and Q respectively, and v_P and v_Q for the vertical components (measured upwards). For the displacements from O, we write x_P and x_Q for the horizontal components and y_P and y_Q for the vertical components.

Resolving horizontally, using the "suvat" equations, we have

$$\begin{aligned} u_P &= V\cos\alpha, \\ u_Q &= kV\cos\beta; \\ x_P &= Vt\cos\alpha, \\ x_Q &= kVt\cos\beta. \end{aligned}$$

Likewise, vertically we have

$$\begin{aligned} v_P &= V\sin\alpha - gt, \\ v_Q &= -kV\sin\beta - gt; \\ y_P &= Vt\sin\alpha - \tfrac{1}{2}gt^2, \\ y_Q &= d - kVt\sin\beta - \tfrac{1}{2}gt^2. \end{aligned}$$

At time T, the particles collide, so $x_P = x_Q$, giving

$$VT\cos\alpha = kVT\cos\beta,$$

so that $\cos\alpha = k\cos\beta$.

Also, $y_P = y_Q$, so
$$VT\sin\alpha - \tfrac{1}{2}gT^2 = d - kVT\sin\beta - \tfrac{1}{2}gT^2,$$
which gives
$$VT\sin\alpha = d - kVT\sin\beta.$$
Now $k^2\sin^2\beta = k^2 - k^2\cos^2\beta = k^2 - \cos^2\alpha$ from the above, so we rearrange and then square both sides of the previous equation to make use of this conclusion:
$$kVT\sin\beta = d - VT\sin\alpha,$$
so
$$k^2V^2T^2\sin^2\beta = (d - VT\sin\alpha)^2,$$
which then gives us
$$V^2T^2(k^2 - \cos^2\alpha) = d^2 - 2dVT\sin\alpha + V^2T^2\sin^2\alpha.$$
Finally, subtracting the right hand side from the left gives
$$V^2T^2(k^2 - \cos^2\alpha - \sin^2\alpha) - d^2 + 2dVT\sin\alpha = 0,$$
or
$$V^2T^2(k^2 - 1) + 2dVT\sin\alpha - d^2 = 0, \qquad (*)$$
as required.

> *Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that*
> $$gd \leqslant (1+k)V^2.$$

At the maximum height, we have $v_P = 0$, so $V\sin\alpha = gT$. Substituting for T in $(*)$ gives us
$$V^2\left(\frac{V\sin\alpha}{g}\right)^2(k^2-1) + 2dV\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - d^2 = 0.$$
Multiplying through by g^2 and expanding brackets gives
$$(k^2-1)V^4\sin^2\alpha + 2gdV^2\sin^2\alpha - g^2d^2 = 0.$$
Thus
$$\sin^2\alpha = \frac{g^2d^2}{(k^2-1)V^4 + 2gdV^2}.$$
For the inequality, the only thing we know for certain is that $\sin^2\alpha \leqslant 1$, and this gives
$$\frac{g^2d^2}{(k^2-1)V^4 + 2gdV^2} \leqslant 1,$$
so
$$g^2d^2 \leqslant (k^2-1)V^4 + 2gdV^2.$$

It is not immediately clear how to continue, so we try completing the square for gd:

$$(gd - V^2)^2 - V^4 \leqslant (k^2 - 1)V^4,$$

so that

$$(gd - V^2)^2 \leqslant k^2 V^4.$$

Now if $a^2 \leqslant b^2$, then $a \leqslant |b|$, so

$$gd - V^2 \leqslant |kV^2|.$$

But $kV^2 > 0$, so we are almost there:

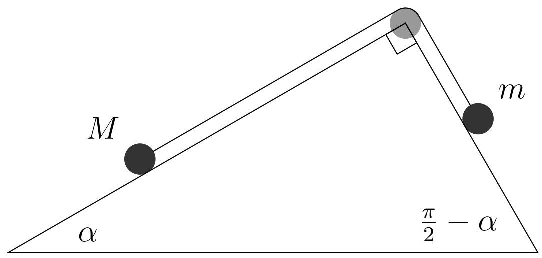
$$gd - V^2 \leqslant kV^2,$$

which finally gives us the required

$$gd \leqslant (1 + k)V^2.$$

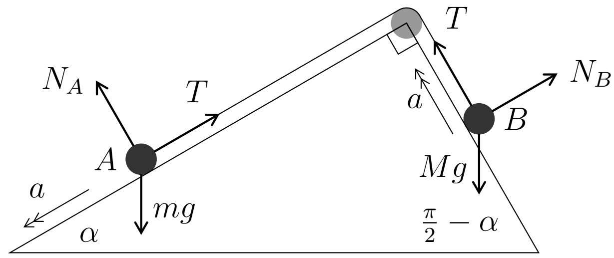
Question 10

A triangular wedge is fixed to a horizontal surface. The base angles of the wedge are α and $\frac{\pi}{2} - \alpha$. Two particles, of masses M and m, lie on different faces of the wedge, and are connected by a light inextensible string which passes over a smooth pulley at the apex of the wedge, as shown in the diagram. The contacts between the particles and the wedge are smooth.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ the particle of mass M will slide down the face of the wedge.

We start by drawing the forces on the picture, and labelling the masses as A (of mass M) and B (of mass m). We let T be the tension in the string.



We now resolve along the faces of the wedge at A and B. (It turns out that resolving normally to the face doesn't help at all for this problem.)

$$\mathscr{R}_A(\nearrow) \qquad T - Mg\sin\alpha = -Ma \tag{1}$$
$$\mathscr{R}_B(\nwarrow) \qquad T - mg\cos\alpha = ma \tag{2}$$

We are not interested in the tension in the string, so we subtract these equations (as $(2)-(1)$) to eliminate T, yielding

$$Mg\sin\alpha - mg\cos\alpha = ma + Ma.$$

Thus, dividing by $M + m$ gives

$$a = \frac{Mg\sin\alpha - mg\cos\alpha}{M+m}. \tag{3}$$

The mass M will slide down the slope if and only if $a > 0$, that is if and only if

$$Mg\sin\alpha - mg\cos\alpha > 0.$$

Rearranging and dividing by $g\cos\alpha$ gives $M\tan\alpha > m$, or

$$\tan\alpha > \frac{m}{M}.$$

(ii) Given that $\tan\alpha = \dfrac{2m}{M}$*, show that the magnitude of the acceleration of the particles is*

$$\frac{g\sin\alpha}{\tan\alpha + 2}$$

and that this is maximised at $4m^3 = M^3$ *.*

We simply substitute $2m/M = \tan\alpha$ into our formula (3) for a to find the magnitude of the acceleration:

$$\begin{aligned}
a &= \frac{Mg\sin\alpha - mg\cos\alpha}{M+m} \\
&= \frac{2g\sin\alpha - (2m/M)g\cos\alpha}{2+(2m/M)} \qquad \text{multiplying by } 2/M \\
&= \frac{2g\sin\alpha - g\tan\alpha\cos\alpha}{2+\tan\alpha} \\
&= \frac{2g\sin\alpha - g\sin\alpha}{2+\tan\alpha} \\
&= \frac{g\sin\alpha}{2+\tan\alpha}
\end{aligned}$$

To maximise this with respect to α, we differentiate with respect to α (using the quotient rule) and solve $\mathrm{d}a/\mathrm{d}\alpha = 0$:

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}\alpha} &= \frac{(2+\tan\alpha)g\cos\alpha - \sec^2\alpha . g\sin\alpha}{(2+\tan\alpha)^2} \\
&= \frac{g(2\cos\alpha + \sin\alpha - \sec^2\alpha\sin\alpha)}{(2+\tan\alpha)^2} \\
&= 0
\end{aligned}$$

so we require

$$2\cos\alpha + \sin\alpha - \sec^2\alpha\sin\alpha = 0.$$

We can simplify this by dividing through by $\cos\alpha$, so that we are able to express everything in terms of $\tan\alpha$:

$$2 + \tan\alpha - \sec^2\alpha\tan\alpha = 0,$$

so

$$2 + \tan\alpha - (1+\tan^2\alpha)\tan\alpha = 0,$$

so that $\tan^3\alpha = 2$. Remembering that $\tan\alpha = 2m/M$, we finally get

$$\frac{8m^3}{M^3} = 2,$$

which leads immediately to $M^3 = 4m^3$.

We finally need to ensure that this stationary point gives us a maximum; this is clear, since as $\alpha \to 0$, $a \to 0$ and as $\alpha \to \frac{\pi}{2}$, $a \to 0$.

Question 11

Two particles move on a smooth horizontal table and collide. The masses of the particles are m and M. Their velocities before the collision are $u\mathbf{i}$ and $v\mathbf{i}$, respectively, where $\mathbf{i}$ is a unit vector and $u > v$. Their velocities after the collision are $p\mathbf{i}$ and $q\mathbf{i}$, respectively. The coefficient of restitution between the two particles is e, where $e < 1$.

(i) Show that the loss of kinetic energy due to the collision is

$$\tfrac{1}{2}m(u-p)(u-v)(1-e),$$

and deduce that $u \geqslant p$.

Before the collision:

After the collision:

Conservation of momentum gives:

$$mu + Mv = mp + Mq. \tag{1}$$

Newton's Law of Restitution gives:

$$q - p = e(u - v). \tag{2}$$

Now the loss of kinetic energy due to the collision is

$$\begin{aligned} E &= \text{initial KE} - \text{final KE} \\ &= (\tfrac{1}{2}mu^2 + \tfrac{1}{2}Mv^2) - (\tfrac{1}{2}mp^2 + \tfrac{1}{2}Mq^2) \\ &= \tfrac{1}{2}m(u^2 - p^2) + \tfrac{1}{2}M(v^2 - q^2) \\ &= \tfrac{1}{2}m(u-p)(u+p) + \tfrac{1}{2}M(v-q)(v+q). \end{aligned}$$

Now from (1), we get $M(v-q) = m(p-u)$, so we get

$$\begin{aligned} E &= \tfrac{1}{2}m(u-p)(u+p) + \tfrac{1}{2}m(p-u)(v+q) \\ &= \tfrac{1}{2}m(u-p)\big((u+p) - (v+q)\big) \\ &= \tfrac{1}{2}m(u-p)\big((u-v) - (q-p)\big) \\ &= \tfrac{1}{2}m(u-p)\big((u-v) - e(u-v)\big) \\ &= \tfrac{1}{2}m(u-p)(u-v)(1-e). \end{aligned}$$

Now, since the loss of energy cannot be negative, we have $E \geqslant 0$. But we are given that $e < 1$ and $u > v$, so we must have $u - p \geqslant 0$, or $u \geqslant p$.

> (ii) *Given that each particle loses the same (non-zero) amount of kinetic energy in the collision, show that*
> $$u + v + p + q = 0,$$
> *and that, if* $m \neq M$,
> $$e = \frac{(M + 3m)u + (3M + m)v}{(M - m)(u - v)}.$$

The first particle loses an amount of kinetic energy equal to $\frac{1}{2}m(u^2 - p^2)$; the second loses $\frac{1}{2}M(v^2 - q^2)$, so we are given

$$\tfrac{1}{2}m(u^2 - p^2) = \tfrac{1}{2}M(v^2 - q^2),$$

so

$$m(u^2 - p^2) - M(v^2 - q^2) = 0.$$

Again, we use $M(v - q) = m(p - u)$, so that

$$\begin{aligned} m(u^2 - p^2) - M(v^2 - q^2) &= m(u + p)(u - p) - M(v + q)(v - q) \\ &= m(u + p)(u - p) - m(v + q)(p - u) \\ &= m(p + q + u + v)(u - p) \\ &= 0. \end{aligned}$$

Since the amount of kinetic energy lost is non-zero, we have $E > 0$ (in the notation of part (i)), so that $u > p$. Thus we must have $p + q + u + v = 0$.

We can also equate the loss of energy of each particle with $\frac{1}{2}E$, so:

$$\tfrac{1}{2}m(u^2 - p^2) = \tfrac{1}{4}m(u - p)(u - v)(1 - e),$$

giving

$$u + p = \tfrac{1}{2}(u - v)(1 - e). \tag{3}$$

We now need to eliminate p, and rearrange to get an expression for e. Equation (2) gives

$$Mq - Mp = Me(u - v),$$

and subtracting this from (1), $mp + Mq = mu + Mv$, gives

$$(m + M)p = (m - Me)u + (M + Me)v.$$

We multiply (3) by $m + M$ to get:

$$(m + M)p + (m + M)u = \tfrac{1}{2}(M + m)(u - v)(1 - e),$$

and then substitute in our expression for $(m + M)p$ to give us:

$$(m - Me)u + (M + Me)v + (m + M)u = \tfrac{1}{2}(M + m)(u - v)(1 - e).$$

We expand and rearrange to collect terms which are multiples of e:

$$2(mu - Mue + Mv + Mve + mu + Mu) = (Mu - Mv + mu - mv)(1 - e)$$

so

$$(2Mv - 2Mu + Mu - Mv + mu - mv)e = Mu - Mv + mu - mv - 4mu - 2Mv - 2Mu,$$

which leads to

$$(M - m)(v - u)e = -Mu - 3Mv - 3mu - mv.$$

We can write the right hand side as $-(M + 3m)u - (3M + m)v$, so that assuming $M \neq m$, we can divide by $(M - m)(v - u)$ to get

$$e = \frac{(M + 3m)u + (3M + m)v}{(M - m)(v - u)},$$

as we wanted.

Question 12

Prove that, for any real numbers x and y, $x^2 + y^2 \geqslant 2xy$.

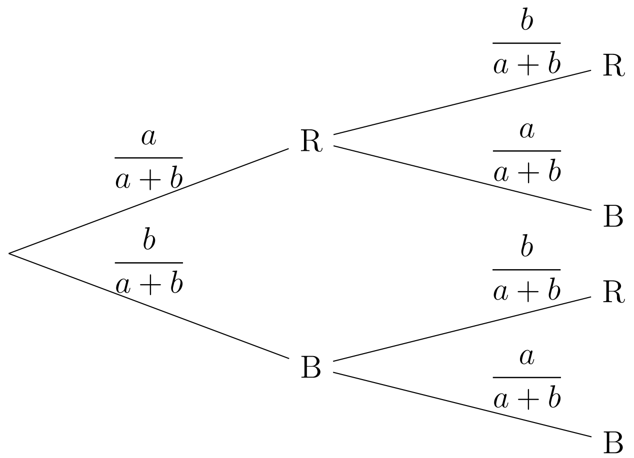
We can rearrange the inequality to get

$$x^2 - 2xy + y^2 \geqslant 0.$$

But the left hand side is just $(x-y)^2$, so this inequality becomes $(x-y)^2 \geqslant 0$. This is clearly true, as any real number squared is non-negative, and since this is just a rearrangement of the desired inequality, that must also be true.

(i) Carol has two bags of sweets. The first bag contains a red sweets and b blue sweets, whereas the second bag contains b red sweets and a blue sweets, where a and b are positive integers. Carol shakes the bags and picks one sweet from each bag without looking. Prove that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours.

We can draw a tree diagram to represent this situation:



So

$$\begin{aligned}
\text{P(same colour)} &= \frac{a}{a+b} \cdot \frac{b}{a+b} + \frac{b}{a+b} \cdot \frac{a}{a+b} \\
&= \frac{2ab}{(a+b)^2} \\
\text{P(different colours)} &= \frac{a}{a+b} \cdot \frac{a}{a+b} + \frac{b}{a+b} \cdot \frac{b}{a+b} \\
&= \frac{a^2+b^2}{(a+b)^2}.
\end{aligned}$$

Since $a^2 + b^2 \geqslant 2ab$, it follows that the probability that the sweets are of the same colour cannot exceed the probability that they are of different colours, as required.

(ii) *Simon has three bags of sweets. The first bag contains a red sweets, b white sweets and c yellow sweets, where a, b and c are positive integers. The second bag contains b red sweets, c white sweets and a yellow sweets. The third bag contains c red sweets, a white sweets and b yellow sweets. Simon shakes the bags and picks one sweet from each bag without looking. Show that the probability that exactly two of the sweets are of the same colour is*

$$\frac{3(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)}{(a+b+c)^3},$$

and find the probability that the sweets are all of the same colour. Deduce that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

We argue in the same way:

$$\begin{aligned}
\mathrm{P}(\text{exactly 2 same}) &= \mathrm{P}(\mathrm{RRW})+\mathrm{P}(\mathrm{RRY})+\mathrm{P}(\mathrm{RWR})+\\
&\qquad \mathrm{P}(\mathrm{RYR})+\mathrm{P}(\mathrm{WRR})+\mathrm{P}(\mathrm{YRR})+\\
&\quad \mathrm{P}(\mathrm{WWR})+\mathrm{P}(\mathrm{WWY})+\mathrm{P}(\mathrm{WRW})+\\
&\qquad \mathrm{P}(\mathrm{WYW})+\mathrm{P}(\mathrm{RWW})+\mathrm{P}(\mathrm{YWW})+\\
&\quad \mathrm{P}(\mathrm{YYR})+\mathrm{P}(\mathrm{YYW})+\mathrm{P}(\mathrm{YRY})+\\
&\qquad \mathrm{P}(\mathrm{YWY})+\mathrm{P}(\mathrm{RYY})+\mathrm{P}(\mathrm{WYY})\\
&= \frac{aba}{(a+b+c)^3}+\frac{abb}{(a+b+c)^3}+\frac{acc}{(a+b+c)^3}+\\
&\qquad \frac{aac}{(a+b+c)^3}+\frac{bbc}{(a+b+c)^3}+\frac{cbc}{(a+b+c)^3}+\\
&\quad \frac{bcc}{(a+b+c)^3}+\frac{bcb}{(a+b+c)^3}+\frac{bba}{(a+b+c)^3}+\\
&\qquad \frac{baa}{(a+b+c)^3}+\frac{aca}{(a+b+c)^3}+\frac{cca}{(a+b+c)^3}+\\
&\quad \frac{cac}{(a+b+c)^3}+\frac{caa}{(a+b+c)^3}+\frac{cbb}{(a+b+c)^3}+\\
&\qquad \frac{ccb}{(a+b+c)^3}+\frac{aab}{(a+b+c)^3}+\frac{bab}{(a+b+c)^3}\\
&= \frac{3(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)}{(a+b+c)^3}.
\end{aligned}$$

More simply, the probability that all three are the same colour is given by

$$\begin{aligned}
\mathrm{P}(\text{all 3 same}) &= \mathrm{P}(\mathrm{RRR})+\mathrm{P}(\mathrm{WWW})+\mathrm{P}(\mathrm{YYY})\\
&= \frac{abc}{(a+b+c)^3}+\frac{bca}{(a+b+c)^3}+\frac{cab}{(a+b+c)^3}\\
&= \frac{3abc}{(a+b+c)^3}.
\end{aligned}$$

Now to find the inequality we want, we apply the initial inequality again: we have $a^2 + b^2 \geqslant 2ab$, so $a^2c + b^2c \geqslant 2abc$. Similarly, $b^2 + c^2 \geqslant 2bc$, so $b^2a + c^2a \geqslant 2abc$, and finally, $c^2b + a^2b \geqslant 2abc$. Thus

$$\begin{aligned}\frac{3(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)}{(a+b+c)^3} &= \frac{3(a^2c + b^2c + b^2a + c^2a + c^2b + a^2b)}{(a+b+c)^3} \\ &\geqslant \frac{3(2abc + 2abc + 2abc)}{(a+b+c)^3} \\ &= \frac{6(3abc)}{(a+b+c)^3},\end{aligned}$$

showing that the probability that exactly two of the sweets are of the same colour is at least 6 times the probability that the sweets are all of the same colour.

Question 13

I seat n boys and 3 girls in a line at random, so that each order of the $n+3$ children is as likely to occur as any other. Let K be the maximum number of consecutive girls in the line so, for example, $K = 1$ if there is at least one boy between each pair of girls.

(i) Find $\mathrm{P}(K = 3)$.

There are two equivalent ways to approach this question: either to regard the boys and girls as all distinct, so that there are $(n + 3)!$ possible orders, or to regard all boys as indistinguishable and all girls as indistinguishable, so that there are $\binom{n+3}{3}$ possible orders. We use the latter approach here.

We note that, regarding the boys as indistinguishable and the girls as indistinguishable, there are

$$\binom{n+3}{3} = \frac{1}{6}(n+3)(n+2)(n+1)$$

possible arrangements of the students.

If $K = 3$, this means that all three girls are adjacent. So the situation must be that there are r boys, followed by 3 girls, followed by $n - r$ boys, where $r = 0, 1, \ldots, n$, so there are $n + 1$ possibilities.

Thus

$$\begin{aligned}\mathrm{P}(K = 3) &= \frac{n+1}{\frac{1}{6}(n+3)(n+2)(n+1)} \\ &= \frac{6}{(n+2)(n+3)}.\end{aligned}$$

(ii) Show that

$$\mathrm{P}(K = 1) = \frac{n(n-1)}{(n+2)(n+3)}.$$

Approach 1: Counting explicitly

To have $K = 1$, we must have each pair of girls separated by at least one boy, like this:

$$\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_1}\,\mathrm{G}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_2}\,\mathrm{G}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_3}\,\mathrm{G}\,\underbrace{\mathrm{B}\ldots\mathrm{B}}_{r_4}$$

where $r_1 \geqslant 0$, $r_2 > 0$, $r_3 > 0$, $r_4 \geqslant 0$ and $r_1 + r_2 + r_3 + r_4 = n$.

If r_1 and r_2 are fixed, then we must have $r_3 + r_4 = n - (r_1 + r_2)$, so we can have $r_3 = 1, 2, \ldots, n - (r_1 + r_2)$, giving $n - (r_1 + r_2)$ possibilities for r_3 and r_4.

Thus, if r_1 is fixed, r_2 could be 1, 2, ..., $n - r_1 - 1$ (but not $n - r_1$, as we must have $r_3 > 0$). Thus the number of possibilities for a fixed value of r_1 is given by

$$\begin{aligned}\sum_{r_2=1}^{n-r_1-1} (n - r_1 - r_2) &= \sum_{r_2=1}^{n-r_1-1} (n - r_1) - \sum_{r_2=1}^{n-r_1-1} r_2 \\ &= (n - r_1 - 1)(n - r_1) - \tfrac{1}{2}(n - r_1 - 1)(n - r_1) \\ &= \tfrac{1}{2}(n - r_1 - 1)(n - r_1) \\ &= \tfrac{1}{2}(n^2 - 2nr_1 + r_1^2 - n + r_1).\end{aligned}$$

Now, r_1 can take the values 0, 1, 2, ..., $n - 2$ (as we need $r_1 > 0$ and $r_2 > 0$), giving the total number of possibilities as

$$\begin{aligned}&\sum_{r_1=0}^{n-2} \tfrac{1}{2}(n^2 - 2nr_1 + r_1^2 - n + r_1) \\ &\quad = \sum_{r_1=0}^{n-2} \tfrac{1}{2}(n^2 - n) - \tfrac{1}{2}(2n - 1)\sum_{r_1=0}^{n-2} r_1 + \tfrac{1}{2}\sum_{r_1=0}^{n-2} r_1^2 \\ &\quad = \tfrac{1}{2}(n - 1)(n^2 - n) - \tfrac{1}{2}(2n - 1) \cdot \tfrac{1}{2}(n - 2)(n - 1) + \tfrac{1}{12}(n - 2)(n - 1)(2n - 3) \\ &\quad = \tfrac{1}{12}(n - 1)\big(6(n^2 - n) - 3(2n - 1)(n - 2) + (n - 2)(2n - 3)\big) \\ &\quad = \tfrac{1}{12}(n - 1)(6n^2 - 6n - 6n^2 + 15n - 6 + 2n^2 - 7n + 6) \\ &\quad = \tfrac{1}{12}(n - 1)(2n^2 + 2n) \\ &\quad = \tfrac{1}{6}n(n - 1)(n + 1).\end{aligned}$$

Thus we can finally deduce

$$\begin{aligned}\mathrm{P}(K = 1) &= \frac{\tfrac{1}{6}n(n - 1)(n + 1)}{\tfrac{1}{6}(n + 3)(n + 2)(n + 1)} \\ &= \frac{n(n - 1)}{(n + 2)(n + 3)}.\end{aligned}$$

Approach 2: A combinatorial argument

We have to place each of the three girls either between two boys or at the end of the line, and we cannot have two girls adjacent. We can think of the line as n boys with gaps between them and at the ends, like this:

$$\_\,\mathrm{B}\,\_\,\mathrm{B}\,\_\,\ldots\,\_\,\mathrm{B}\,\_\,\mathrm{B}\,\_$$

Note that there are $n + 1$ gaps (one to the right of each boy, and one at the left of the line). Three of the gaps are to be filled with girls, giving $\binom{n+1}{3} = \frac{1}{6}(n+1)n(n-1)$ ways of choosing them. Therefore

$$\begin{aligned}\mathrm{P}(K = 1) &= \frac{\tfrac{1}{6}(n + 1)n(n - 1)}{\tfrac{1}{6}(n + 3)(n + 2)(n + 1)} \\ &= \frac{n(n - 1)}{(n + 2)(n + 3)}.\end{aligned}$$

Approach 3: Another combinatorial argument

We add one more boy at the right end of the line. In this way, we have a boy to the right of each girl, as follows:

$$\underbrace{\text{B}\ldots\text{B}}_{r_1}\,\text{GB}\,\underbrace{\text{B}\ldots\text{B}}_{r_2}\,\text{GB}\,\underbrace{\text{B}\ldots\text{B}}_{r_3}\,\text{GB}\,\underbrace{\text{B}\ldots\text{B}}_{r_4}$$

where this time, we have $r_1 \geqslant 0$, $r_2 \geqslant 0$, $r_3 \geqslant 0$ and $r_4 \geqslant 0$. Also, as there are now $n+1$ boys, we have $r_1 + r_2 + r_3 + r_4 = n + 1 - 3 = n - 2$.

So if we think of GB as one 'person', there are $n-2$ boys and 3 GBs, giving $n+1$ 'people' in total. There are $\binom{n+1}{3} = \frac{1}{6}(n+1)n(n-1)$ ways of arranging them, giving

$$\begin{aligned} \mathrm{P}(K=1) &= \frac{\frac{1}{6}(n+1)n(n-1)}{\frac{1}{6}(n+3)(n+2)(n+1)} \\ &= \frac{n(n-1)}{(n+2)(n+3)}. \end{aligned}$$

(iii) Find $\mathrm{E}(K)$.

We could attempt to determine $\mathrm{P}(K=2)$ directly, but it is far easier to note that K can only take the values 1, 2 or 3. Thus

$$\begin{aligned} \mathrm{P}(K=2) &= 1 - \mathrm{P}(K=1) - \mathrm{P}(K=3) \\ &= \frac{(n+2)(n+3) - n(n-1) - 6}{(n+2)(n+3)} \\ &= \frac{n^2 + 5n + 6 - n^2 + n - 6}{(n+2)(n+3)} \\ &= \frac{6n}{(n+2)(n+3)}. \end{aligned}$$

Since $\mathrm{E}(K) = \sum_k k.\mathrm{P}(K=k)$, we can now calculate $\mathrm{E}(K)$:

$$\begin{aligned} \mathrm{E}(K) &= 1.\mathrm{P}(K=1) + 2.\mathrm{P}(K=2) + 3.\mathrm{P}(K=3) \\ &= \frac{n(n-1) + 2\cdot 6n + 3\cdot 6}{(n+2)(n+3)} \\ &= \frac{n^2 - n + 12n + 18}{(n+2)(n+3)} \\ &= \frac{n^2 + 11n + 18}{(n+2)(n+3)} \\ &= \frac{(n+2)(n+9)}{(n+2)(n+3)} \\ &= \frac{n+9}{n+3}. \end{aligned}$$

STEP II, Solutions
2009

1 Both graphs are symmetric in the lines $y = \pm x$, and $x^4 + y^4 = u$ is also symmetric in the x- and y-axes. These facts immediately enable us to write down the coordinates of $B(\beta, \alpha)$, $C(-\alpha, -\beta)$ and $D(-\beta, -\alpha)$. Remember to keep the cyclic order A, B, C, D correct, else this could lead to silly calculational errors later on. The easiest way to show that $ABCD$ is a rectangle is to work out the gradients of the four sides (which turn out to be either 1 or –1) and then note that each pair of adjacent sides is perpendicular using the "product of gradients = –1" result. Working with distances is also a possible solution-approach but, on its own, only establishes that the quadrilateral is a parallelogram. However, the next part requires you to calculate distances anyhow, and we find that CB, DA have length $(\alpha + \beta)\sqrt{2}$ while BA, DC are of length $(\alpha - \beta)\sqrt{2}$. Multiplying these then give the area of $ABCD$ as $2(\alpha^2 - \beta^2)$.

All of this is very straightforward, and the only tricky bit of work comes next. It is important to think of α and β as particular values of x and y satisfying each of the two original equations. It is then clear that $(\alpha^2 - \beta^2)^2 = \alpha^4 + \beta^4 - 2(\alpha^2\beta^2) = u - 2v^2$, so that Area $ABCD = 2\sqrt{u - 2v^2}$. Substituting $u = 81$, $v = 4$ into this formula then gives Area $= 2\sqrt{81 - 2 \times 16} = 14$, which is intended principally as a means of checking that your answer is correct.

2(i) It is perfectly possible to differentiate a^(sin[$\pi\,\mathrm{e}^x$]) by using the *Chain Rule* (on a function of a function of a function) but simplest to take logs. and use implicit differentiation. Then, setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and noting that $\pi\,\mathrm{e}^x$ and $\ln a$ are non-zero, we are left solving the eqn. $\cos(\pi\,\mathrm{e}^x) = 0$ for the turning points. This gives $\mathrm{e}^x = (2n + 1)\tfrac{1}{2}\pi \Rightarrow x = \ln\left(n + \tfrac{1}{2}\right)$, $y = a$ or $\tfrac{1}{a}$, depending upon whether n is even or odd. Although not actually required at this point, it may be helpful to note at this stage that the evens give maxima while the odds give minima. There is, however, a much more obvious approach to finding the TPs that doesn't require differentiation at all, and that is to use what should be well-known properties of the sine function: namely, that $y = a^{\sin(\pi.\exp x)}$ has maxima when $\sin(\pi\,\mathrm{e}^x) = 1$, i.e. $\pi\,\mathrm{e}^x = (2n + \tfrac{1}{2})\pi$, and $x = \ln(2n + \tfrac{1}{2})$ for $n = 0, 1, \dots$, with $y_{\max} = a$. Similarly, minima occur when $\sin(\pi\,\mathrm{e}^x) = -1$, i.e. $\pi\,\mathrm{e}^x = (2n - \tfrac{1}{2})\pi$, and $x = \ln(2n - \tfrac{1}{2})$ for $n = 1, 2, \dots$, with $y_{\min} = \tfrac{1}{a}$.

(ii) Using the addition formula for $\sin(A + B)$, and the approximations given, we have

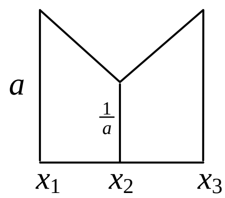
$$\sin(\pi\,\mathrm{e}^x) \approx \sin(\pi + \pi x) = -\sin(\pi x) \approx -\pi x \text{ for small } x,$$

leading to $y \approx a^{-\pi x} = e^{-\pi x \ln a} \approx 1 - \pi x.\ln a$.

(iii) Firstly, we can note that, for $x < 0$, the curve has an asymptote $y = 1$ (as $x \to -\infty$, $y \to 1+$). Next, for $x > 0$, the curve oscillates between a and $\tfrac{1}{a}$, with the peaks and troughs getting ever closer together. The work in (i) helps us identify the TPs: the first max. occurs when $n = 0$ at a *negative* value of x [N.B. $\ln\left(\tfrac{1}{2}\right) < 0$] at $y = a$; while the result in (ii) tells us that the curve is approximately negative linear as it crosses the y-axis.

(iv) The final part provides the only really tricky part to the question, and a quick diagram might be immensely useful here. Noting the relevant x-coordinates

$$x_1 = \ln\left(2k - \tfrac{3}{2}\right),\ x_2 = \ln\left(2k - \tfrac{1}{2}\right), \text{ and } x_3 = \ln\left(2k + \tfrac{1}{2}\right),$$



the area is the sum of two trapezia (or rectangle – triangle), and manipulating $\ln\left(\dfrac{4k+1}{4k-3}\right) = \ln\left(\dfrac{4k-3+4}{4k-3}\right) = \ln\left(1 + \dfrac{1}{k - \frac{3}{4}}\right)$ leads to the final, given answer.

3 Using the "addition" formula for $\tan(A - B)$,

$$\text{LHS} \equiv \tan\left(\tfrac{\pi}{4} - \tfrac{x}{2}\right) \equiv \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} \equiv \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} \equiv \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} \times \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}$$

$$\equiv \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \text{ (since } c^2 + s^2 = 1) \equiv \frac{1 - \sin x}{\cos x} \equiv \sec x - \tan x \equiv \text{RHS.}$$

Alternatively, one could use the "$t = \tan(\frac{1}{2}$-angle)" formulae to show that

$$\text{RHS} \equiv \frac{1 - t^2}{1 + t^2} - \frac{2t}{1 - t^2} \equiv \frac{(1 - t)^2}{(1 - t)(1 + t)} \equiv \frac{1 - t}{1 + t} \equiv \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} \equiv \text{LHS.}$$

(i) Setting $x = \frac{\pi}{4}$ in (*) $\Rightarrow \tan\frac{\pi}{8} = \sqrt{2} - 1$. Then, using the addition formula for $\tan(A + B)$ with $A = \frac{\pi}{3}$ and $B = \frac{\pi}{8}$, we have $\tan\frac{11\pi}{24} = \tan\left(\frac{\pi}{3} + \frac{\pi}{8}\right) = \dfrac{\sqrt{3} + \sqrt{2} - 1}{1 - \sqrt{3}\left(\sqrt{2} - 1\right)} = \dfrac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1}$, as required.

(ii) Now, in the "spirit" of maths, one might reasonably expect that one should take the given expression, rationalise the denominator (twice) and derive the given answer, along the lines …

$$\frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} = \frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} \times \frac{1 + \sqrt{3} + \sqrt{6}}{1 + \sqrt{3} + \sqrt{6}} = \frac{1 + 2\sqrt{2} + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}.$$

However, with a given answer, it is perfectly legitimate merely to multiply across and verify that $\left(\sqrt{3} - \sqrt{6} + 1\right)\left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right) = \sqrt{3} + \sqrt{2} - 1$.

(iii) Having got this far, the end is really very clearly signposted. Setting $x = \frac{11\pi}{24}$ in (*) gives

$$\tan\frac{\pi}{48} = \sec\frac{11\pi}{24} - \tan\frac{11\pi}{24} = \sqrt{1 + t^2} - t$$

$$= \sqrt{1 + \left[4 + 2 + 3 + 6 + 4\sqrt{2} + 4\sqrt{3} + 4\sqrt{6} + 2\sqrt{6} + 2\sqrt{12} + 2\sqrt{18}\right]} - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)$$

$$= \sqrt{15 + 10\sqrt{2} + 8\sqrt{3} + 6\sqrt{6}} - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)$$

4(i) Writing $p(x) - 1 \equiv q(x).(x - 1)^5$, where $q(x)$ is a quartic polynomial, immediately gives $p(1) = 1$.

(ii) Diff[g]. using the product and chain rules leads to

$$p'(x) \equiv q(x).5(x - 1)^4 + q'(x).(x - 1)^5 \equiv (x - 1)^4.\{5\,q(x) + (x - 1)\,q'(x)\},$$

so that $p'(x)$ is divisible by $(x - 1)^4$.

(iii) Similarly, we have that $p'(x)$ is divisible by $(x + 1)^4$ and $p(-1) = -1$.
Thus $p'(x)$ is divisible by $(x + 1)^4.(x - 1)^4 \equiv (x^2 - 1)^4$. However, $p'(x)$ *is* a polynomial of degree eight, hence $p'(x) \equiv k(x^2 - 1)^4$ for some constant k. That is, $p'(x) \equiv k\left(x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right)$.
Integrating term by term then gives $p(x) \equiv k\left(\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right) + C$, and use of both $p(1) = 1$ and $p(-1) = -1$ help to find k and C; namely, $k = \dfrac{315}{128}$ and $C = 0$.

5 The very first bit is not just a giveaway mark, but rather a helpful indicator of the kind of result or technique that may be used in this question: $\left(\sqrt{x-1}+1\right)^2 = x + 2\sqrt{x-1}$; but pay attention to what happens here. Most particularly, the fact that $\left(\sqrt{x-1}+1\right)^2 = x + 2\sqrt{x-1}$ does NOT necessarily mean that $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ since positive numbers have *two* square-roots! Recall that $\sqrt{x^2} = |x|$ and not just *x*. Notice that, during the course of this question, the range of values under consideration switches from (5, 10) to $\left(\frac{5}{4}, 10\right)$, and one doesn't need to be particularly suspicious to wonder why this is so. A modicum of investigation at the outset seems warranted here, as to when things change sign.

(i) So … while $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ seems a perfectly acceptable thing to write, since $x \ge 1$ is a necessary condition in order to be able to take square-roots at all here (for real numbers), simply writing down that $\sqrt{x-2\sqrt{x-1}} = +\left(\sqrt{x-1}-1\right)$ *may* cause a problem. A tiny amount of exploration shows that $\sqrt{x-1}-1$ changes from negative to positive around $x = 2$. Hence, in part (i), we can ignore any negative considerations and plough ahead: $I = \int_5^{10} 2\,\mathrm{d}x = \left[2x\right]_5^{10} = 10$.

(ii) Here in (ii), however, you should realise that the area requested is the sum of two portions, one of which lies below the *x*-axis, and would thus contribute negatively to the total if you failed to take this into account. Thus,

$$\text{Area} = \int_{1.25}^{2} \frac{1-\sqrt{x-1}}{\sqrt{x-1}}\,\mathrm{d}x + \int_{2}^{10} \frac{\sqrt{x-1}-1}{\sqrt{x-1}}\,\mathrm{d}x = \int_{1.25}^{2}\left[(x-1)^{-\frac{1}{2}}-1\right]\mathrm{d}x + \int_{2}^{10}\left[1-(x-1)^{-\frac{1}{2}}\right]\mathrm{d}x$$

$$= \left[2\sqrt{x-1}-x\right]_{1.25}^{2} + \left[x-2\sqrt{x-1}\right]_{2}^{10} = 4\tfrac{1}{4}\,.$$

(iii) Now $\left(\sqrt{x+1}-1\right)^2 = x+2-2\sqrt{x+1}\ \ \forall\, x \ge 0$ so we have no cause for concern here. Then

$$I = \int_{x=1.25}^{10} \frac{1+\sqrt{x-1}+\sqrt{x+1}-1}{\sqrt{x-1}\sqrt{x+1}}\,\mathrm{d}x = \int_{x=1.25}^{10}\left((x+1)^{-\frac{1}{2}}+(x-1)^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$= \left[2\sqrt{x+1}+2\sqrt{x-1}\right]_{1.25}^{10} = 2\left(\sqrt{11}+1\right)$$

6 If you don't know about the *Fibonacci Numbers* by now, then … shame on you! Nevertheless, the first couple of marks for writing down the next few terms must count as among the easiest on the paper. ($F_1 = 1$, $F_2 = 1$), $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$ and $F_{10} = 55$.

(i) If you're careful, the next section isn't particularly difficult either. Using the recurrence relation gives $\frac{1}{F_i} = \frac{1}{F_{i-1}+F_{i-2}} > \frac{1}{2F_{i-1}}$ since $F_{i-2} < F_{i-1}$ for $i \ge 4$. Splitting off the first few terms then leads to $S = \sum_{i=1}^{n} \frac{1}{F_i} > \frac{1}{F_1} + \frac{1}{F_2}\left(1+\frac{1}{2}+\frac{1}{4}+.....\right)$ or $\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_3}\left(1+\frac{1}{2}+\frac{1}{4}+.....\right)$, where the long bracket at the end is the sum-to-infinity of a GP. These give, respectively, $S > 1 + 1 \times 2 = 3$ or $1 + 1 + \frac{1}{2}\times 2 = 3$. A simpler approach could involve nothing more complicated than adding the terms until a sum greater than 3 is reached, which happens when you reach F_5.

A similar approach yields $\frac{1}{F_i} < \frac{1}{2}\left(\frac{1}{F_{i-2}}\right)$ for $i \ge 3$ and splitting off the first few terms, this time separating the odd- and even-numbered terms, gives

$$S = \sum_{i=1}^{n} \frac{1}{F_i} = \frac{1}{F_1} + \frac{1}{F_2} + \left(\frac{1}{F_3} + \frac{1}{F_5} + ...\right) + \left(\frac{1}{F_4} + \frac{1}{F_6} + ...\right)$$

$$< 1 + 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right)$$

$$= 1 + 1 + \frac{1}{2} \times 2 + \frac{1}{3} \times 2 = 3\tfrac{2}{3}.$$

(ii) To show that $S > 3.2$, we simply apply the same approaches as before, but taking more terms initially before summing our GP (or stopping at F_7 in the "simpler approach" mentioned previously). Something like

$$S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 = 3\tfrac{7}{30} > 3\tfrac{6}{30} = 3.2$$

does the job pretty readily. Then, to show that $S < 3\tfrac{1}{2}$, a similar argument to those you have been directed towards by the question, works well with little extra thought required:

$$S < 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) + \frac{1}{8}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right)$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 + \frac{1}{8} \times 2 = 3\tfrac{29}{60} < 3\tfrac{1}{2}.$$

Returning to the initial argument, $F_i < 2\,F_{i-1}$ or $\frac{1}{F_i} > \frac{1}{2}\left(\frac{1}{F_{i-1}}\right)$ for $i \ge 4$, we can extend this to $F_i > \frac{3}{2}\,F_{i-1}$ or $\frac{1}{F_i} < \frac{2}{3}\left(\frac{1}{F_{i-1}}\right)$ for $i \ge 5$, $F_i < \frac{5}{3}\,F_{i-1}$ or $\frac{1}{F_i} > \frac{3}{5}\left(\frac{1}{F_{i-1}}\right)$ for $i \ge 6$, etc., simply by using the defining recurrence relation for the *Fibonacci Numbers*, leading to the general results

$$F_n > \left(\frac{F_{2k}}{F_{2k-1}}\right) F_{n-1} \quad \text{or} \quad \frac{1}{F_n} < \left(\frac{F_{2k-1}}{F_{2k}}\right)\frac{1}{F_{n-1}} \quad \text{for } n \ge 2k+1$$

and

$$F_n < \left(\frac{F_{2k+1}}{F_{2k}}\right) F_{n-1} \quad \text{or} \quad \frac{1}{F_n} > \left(\frac{F_{2k}}{F_{2k+1}}\right)\frac{1}{F_{n-1}} \quad \text{for } n \ge 2k+2.$$

Since the terms $\frac{F_n}{F_{n-1}} \to \phi = \frac{\sqrt{5}+1}{2}$, the golden ratio, (being the positive root of the quadratic equation $x^2 = x + 1$, we can deduce the approximation $S \approx \sum_{i=1}^{n} \frac{1}{F_i} + \frac{1}{F_{n+1}}\phi^2$ since the geometric progression $1 + \frac{1}{\phi} + \frac{1}{\phi^2} + ... = \frac{1}{1-\frac{1}{\phi}} = \frac{\phi}{\phi - 1} = \frac{\phi}{\frac{1}{\phi}} = \phi^2$. Taking $n = 9$, (i.e. just using the first 10 *Fibonacci Numbers* which you were led to write down at the start),

$$S \approx \sum_{i=1}^{9} \frac{1}{F_i} + \frac{1}{F_{10}}\phi^2 = \frac{614893}{185640} + \frac{1}{55} \times \frac{\sqrt{5}+3}{2} \approx 3.359\,89,$$

which is correct to 5 d.p. For further information on this number, try looking up the '*Reciprocal Fibonacci constant*' on Wikipedia, for instance.

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7 It is easy to saunter into this question's opening without pausing momentarily to wonder if one is going about it in the best way. Whilst many can cope with differentiating a "triple"-product with ease, many others can't. However, even for interests' sake, one might stop to consider a general approach to such matters. Differentiating $y = pqr$ (all implicitly functions of x) as, initially, $p(qr)$ and applying the product-rule twice, one obtains $y' = pq\,r' + p\,q'\,r + p'\,qr$, and this can be used here with $p = (x - a)^n$, $q = e^{bx}$ and $r = \sqrt{1+x^2}$ without the need for a lot of the mess (and subsequent mistakes) that was (were) made by so many candidates. Here, $y = (x - a)^n\, e^{bx}\sqrt{1+x^2}$ gives

$$\frac{dy}{dx} = (x-a)^n\, e^{bx}\frac{x}{\sqrt{1+x^2}} + (x-a)^n\, b\, e^{bx}\sqrt{1+x^2} + n(x-a)^{n-1}\, e^{bx}\sqrt{1+x^2}$$

Factorising out the given terms $\Rightarrow \dfrac{(x-a)^{n-1}\,e^{bx}}{\sqrt{1+x^2}}\left\{x(x-a) + b(x-a)(1+x^2) + n(1+x^2)\right\}$, and we are only required to note that the term in the brackets is, indeed, a cubic; though it may prove helpful later on to simplify it by multiplying out and collecting up terms, to get

$$q(x) = bx^3 + (n + 1 - ab)x^2 + (b - a)x + (n - ab).$$

(i) The first integral, $I_1 = \displaystyle\int \frac{(x-4)^{14}\,e^{4x}}{\sqrt{1+x^2}}\left(4x^3 - 1\right)\,dx$, might reasonably be expected to be a very straightforward application of the general result, and so it proves to be. With $n = 15$, and taking $a = b = 4$, so that $q(x) = 4x^3 - 1$ (which really should be checked explicitly), we find

$$I_1 = (x-4)^{15}\, e^{4x}\sqrt{1+x^2} \quad (+\, C).$$

(ii) This second integral, $I_2 = \displaystyle\int \frac{(x-1)^{21}\,e^{12x}}{\sqrt{1+x^2}}\left(12x^4 - x^2 - 11\right)\,dx$, is clearly not so straightforward, since the bracketed term is now quartic. Of the many things one *might* try, however, surely the simplest is to try to factor out a linear term, the obvious candidate being $(x - 1)$.

Finding that $12x^4 - x^2 - 11 \equiv (x - 1)(12x^3 + 12x^2 + 11x + 11)$, we now try $n = 23$, $a = 1$, $b = 12$ to obtain $q(x) = 12x^3 + 12x^2 + 11x + 11$ and $I_2 = (x - 1)^{23}\, e^{12x}\sqrt{1+x^2}$ $(+\, C)$.

(iii) The final integral, $I_3 = \displaystyle\int \frac{(x-2)^{6}\,e^{4x}}{\sqrt{1+x^2}}\left(4x^4 + x^3 - 2\right)\,dx$, is clearly intended to be even less simple than its predecessor. However, you might now suspect that "the next case up" is in there somewhere. So, if you try $n = 8$, $a = 2$, $b = 4$, which gives

$$\frac{dy_8}{dx} = \frac{(x-2)^7\,e^{4x}}{\sqrt{1+x^2}}\left\{4x^3 + x^2 + 2x\right\} = \frac{(x-2)^6\,e^{4x}}{\sqrt{1+x^2}}\left\{4x^4 - 7x^3 - 4x\right\},$$

as well as the obvious target $n = 7$, $a = 2$, $b = 4$, which yields

$$\frac{dy_7}{dx} = \frac{(x-2)^6\,e^{4x}}{\sqrt{1+x^2}}\left\{4x^3 + 2x - 1\right\},$$

It may now be clear that ***both*** are involved. Indeed,

$$I_3 = \int\left(\frac{dy_8}{dx} + 2\frac{dy_7}{dx}\right)dx = y_8 + 2\,y_7 = x(x-2)^7\, e^{4x}\sqrt{1+x^2} \quad (+\, C).$$

8 For the diagram, you are simply required to show P on AB, strictly between A and B; and Q on AC on other side of A to C. The two given parameters indicate that $CQ = \mu\, AC$ and $BP = \lambda\, AB$. Substituting these into the given expression, $CQ \times BP = AB \times AC \Rightarrow \mu AC \,.\, \lambda AB = AB \,.\, AC$ $\Rightarrow \mu = \frac{1}{\lambda}$. [Notice that CQ, BP, etc., are scalar quantities, and hence the "×" **cannot** be the vector product!]

Writing the equation of line PQ in the form $\mathbf{r} = t\,\mathbf{p} + (1-t)\,\mathbf{q}$ for some scalar parameter t and substituting the given forms for $\mathbf{p}$ and $\mathbf{q}$ gives $\mathbf{r} = t\lambda\,\mathbf{a} + t(1-\lambda)\mathbf{b} + (1-t)\mu\,\mathbf{a} + (1-t)(1-\mu)\mathbf{c}$. Eliminating $\mu = \frac{1}{\lambda} \Rightarrow \mathbf{r} = \left(t\lambda + \frac{1}{\lambda} - \frac{t}{\lambda}\right)\mathbf{a} + t(1-\lambda)\mathbf{b} + (1-t)\left(\frac{\lambda-1}{\lambda}\right)\mathbf{c}$. Comparing this to the given answer, we note that when $t = \frac{1}{1-\lambda}$ from the $\mathbf{b}$-component, $1 - t = \frac{\lambda}{\lambda - 1}$, etc., so that we do indeed get $\mathbf{r} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$, as required.

Since $\mathbf{d} - \mathbf{c} = \mathbf{b} - \mathbf{a}$, one pair of sides of opposite sides of $ABDC$ are equal and parallel, so we can conclude that $ABDC$ is a parallelogram

9 (i) If you "break the lamina up" into a rectangle and a triangle (shapes whose geometric centres should be well-known to you), with relative masses 2 and 1, and impose (mentally, at least) a coordinate system onto the diagram, then the x-coordinate of the centre of mass is given by

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{2 \times \frac{9}{2} + 1 \times 12}{3} = 7.$$

(ii) A more detailed approach, but still along similar lines, might be constructed in the following, tabular way:

Shape	Mass	Dist. c.o.m. from OZ
LH end	540ρ	7
RH end	540ρ	7
Front	$41d\rho$	$\frac{27}{2}$
Back	$40d\rho$	0
Base	$9d\rho$	$\frac{9}{2}$

Note that each mass has been calculated as area × density (ρ)

Then $x_E = \dfrac{2 \times (540\rho) \times 7 + 41d\rho \times \frac{27}{2} + 0 + 9dp \times \frac{9}{2}}{1080\rho + 90d\rho}$, which (after much cancelling) simplifies to

$$= \frac{2 \times 60 \times 7 + 66d}{10(12+d)} = \frac{3(140 + 11d)}{5(12+d)}.$$

A similar approach for the full tank gives

Object	Mass	Dist. c.o.m. from OZ
Tank	2880ρ	$\frac{27}{4}$
Water	$10800k\rho$	7

and $x_F = \dfrac{2880\rho \times \frac{27}{4} + 10800kp \times 7}{2880\rho + 10800k\rho} = \dfrac{27 + 105k}{4 + 15k}$.

10 The standard approach in collision questions is to write down the equations gained when applying the principles of *Conservation of Linear Momentum* (CLM) and *Newton's Experimental Law of Restitution* (NEL or NLR), and then what can be deduced from these.

For $P_{1,\,2}$: CLM $\Rightarrow m_1 u = m_1 v_1 + m_2 v_2$ and NEL $\Rightarrow eu = v_2 - v_1$.
Solving to determine the final speeds of P_1 and P_2 then yields

$$v_1 = \frac{(m_1 - em_2)}{m_1 + m_2}u \text{ and } v_2 = \frac{m_1(1+e)}{m_1 + m_2}u\,.$$

Similarly, for $P_{4,\,3}$: CLM $\Rightarrow m_4 u = m_4 v_4 + m_3 v_3$ and NEL $\Rightarrow eu = v_3 - v_4$, leading to

$$v_3 = \frac{m_4(1+e)}{m_3 + m_4}u \text{ and } v_4 = \frac{(m_4 - em_3)}{m_3 + m_4}u\,.$$

If we now write $X = OP_2$ and $Y = OP_3$ initially, and equate the times to the following collisions at O, we have

(1st collision): $\dfrac{(m_1 + m_2)X}{m_1(1+e)u} = \dfrac{(m_3 + m_4)Y}{m_4(1+e)u}$

and

(2nd collision): $\dfrac{(m_1 + m_2)X}{(m_1 - em_2)u} = \dfrac{(m_3 + m_4)Y}{(m_4 - em_3)u}$.

Cancelling u's and $(1 + e)$'s

$$\Rightarrow \frac{(m_1 + m_2)X}{m_1} = \frac{(m_3 + m_4)Y}{m_4} \quad \text{and} \quad \frac{(m_1 + m_2)X}{(m_1 - em_2)} = \frac{(m_3 + m_4)Y}{(m_4 - em_3)}\,. \qquad \textbf{(*)}$$

Dividing these two (or equating for X / Y) $\Rightarrow \dfrac{m_1 - em_2}{m_1} = \dfrac{m_4 - em_3}{m_4}$, which simplifies to

$\dfrac{m_2}{m_1} = \dfrac{m_3}{m_4}$. Finally substituting back into one of the equations **(*)** then gives

$$X\left(1 + \frac{m_2}{m_1}\right) = Y\left(1 + \frac{m_3}{m_4}\right) \Rightarrow X = Y\,.$$

Rather surprisingly, however, the momentum equations turn out to be totally unnecessary here. Consider …

Collision $P_{1,\,2}$: NEL $\Rightarrow eu = v_2 - v_1$

Collision $P_{4,\,3}$: NEL $\Rightarrow eu = v_3 - v_4$ so that $v_2 - v_1 = v_3 - v_4$ **(*)**.

Next, the two equated sets of times are $\dfrac{X}{v_2} = \dfrac{Y}{v_3}$ and $\dfrac{X}{v_1} = \dfrac{Y}{v_4} \Rightarrow X v_3 = Y v_2$ and $X v_4 = Y v_1$.

Subtracting: $X(v_3 - v_4) = Y(v_2 - v_1) \Rightarrow X = Y$ from **(*)**.

11 N2L $\Rightarrow F_T - (n+1)R = (n+1)Ma$, where F_T is the tractive, or driving, force of the engine.

Using $P = F_T \,.\, v$ then gives $a = \dfrac{\frac{P}{v} - (n+1)R}{M(n+1)}$ or $\dfrac{P-(n+1)Rv}{M(n+1)v}$. Note here that, for $a > 0$ we require $P > (n+1)Rv$.

Writing $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$ gives $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{P-(n+1)Rv}{M(n+1)v}$ which is a "variables separable" first-order differential equation: $\dfrac{M(n+1)v}{P-(n+1)Rv}\,\mathrm{d}v = \mathrm{d}t \Rightarrow \displaystyle\int_0^V \frac{M(n+1)v}{P-(n+1)Rv}\,\mathrm{d}v = \int_0^T 1.\mathrm{d}t \quad (=T)$.

Some care is needed to integrate the LHS here, and the simplest approach is to use a substitution such as $s = P-(n+1)Rv$, $\mathrm{d}s = -R(n+1)\,\mathrm{d}v$ to get

$$T = \frac{M}{R}\int \frac{P-s}{s} \times \frac{\mathrm{d}s}{-R(n+1)} = \frac{-M}{(n+1)R^2}\int\left(\frac{P}{s}-1\right)\mathrm{d}s = \frac{-M}{(n+1)R^2}\big[P\ln(s) - s\big]$$

$$= \frac{-M}{(n+1)R^2}\big[P\ln\big(P-(n+1)Rv\big) - \big(P-(n+1)Rv\big)\big]_0^V$$

$$= \frac{-MP}{(n+1)R^2}\left\{\ln\big(P-(n+1)Rv\big) - P + (n+1)Rv - P\ln P + P - 0\right\}$$

$$= \frac{-MP}{(n+1)R^2}\ln\left(\frac{P-(n+1)Rv}{P}\right) - \frac{MV}{R}$$

More careful algebra is still required to manipulate this into a form in which the given approximation can be used:

$$T = \frac{-MP}{(n+1)R^2}\ln\left(1 - \frac{(n+1)Rv}{P}\right) - \frac{MV}{R}$$

$$\approx \frac{-MP}{(n+1)R^2}\left(-\frac{(n+1)Rv}{P} - \frac{1}{2}\left(\frac{(n+1)Rv}{P}\right)^2 \ldots\ldots\right) - \frac{MV}{R}$$

$$= \frac{MV}{R} + \frac{(n+1)MV^2}{2P}\ldots\ldots - \frac{MV}{R}$$

so that $PT \approx \frac{1}{2}(n+1)MV^2$, and this is just the statement of the *Work-Energy Principle*, namely "Work Done = Change in (Kinetic) Energy", in the case when $R = 0$.

When $R \neq 0$, WD against R = WD by engine – Gain in KE $\Rightarrow$ $(n+1)RX = PT - \frac{1}{2}(n+1)MV^2$.
[Unfortunately, a last-minute change to the wording of the question led to the omission of one of the $(n + 1)$s.]

12 (i) This whole question is something of a "one-trick" game, I'm afraid, and relies heavily on being able to spot that X is just half of a normal distribution. The *Standard Normal Distribution* N(0, 1) is given by $\text{P}(X \le x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} \text{e}^{-\frac{1}{2}t^2}\,\text{d}t$. Once the connection has been spotted, the accompanying pure maths work is fairly simple, including the sketch of the graph. This is particularly important since the function e^{kx^2} cannot be integrated analytically.

(ii) Substituting $t = 2x$, $\text{d}t = 2\,\text{d}x$ and equating to $\dfrac{1}{2}$ (being just the positive half of a normal), gives

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty} \text{e}^{-\frac{1}{2}t^2}\,\text{d}t = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} 2\text{e}^{-2x^2}\,\text{d}x = \frac{1}{2} \Rightarrow \int_0^{\infty} \text{e}^{-2x^2}\text{d}x = \frac{\sqrt{2\pi}}{4}.$$

Since total probability = 1, we have $\dfrac{1}{k} = \dfrac{\sqrt{2\pi}}{4}$ and $k = \dfrac{4}{\sqrt{2\pi}}$.

(iii) Thereafter, $\text{E}(X) = k\displaystyle\int_0^{\infty} x\text{e}^{-2x^2}\text{d}x = k\left[-\frac{1}{4}\text{e}^{-2x^2}\right]_0^{\infty} = \frac{1}{4}k = \frac{1}{\sqrt{2\pi}}$.

Also, $\text{E}(X^2) = k\displaystyle\int_0^{\infty} x \times x\,\text{e}^{-2x^2}\text{d}x = k\left\{\left[-\frac{1}{4}x\text{e}^{-2x^2}\right]_0^{\infty} + \int_0^{\infty}\frac{1}{4}\text{e}^{-2x^2}\text{d}x\right\}$ using integration by parts

$$= k\left\{0 + \frac{1}{4} \times \frac{\sqrt{2\pi}}{4}\right\} = \frac{1}{4}.$$

Then $\text{Var}(X) = \text{E}(X^2) - E^2(X) = \dfrac{1}{4} - \dfrac{1}{2\pi}$ or $\dfrac{\pi - 2}{4\pi}$.

(iv) For the median, we want to find the value m of x for which $\dfrac{1}{2} = \dfrac{4}{\sqrt{2\pi}}\displaystyle\int_0^{m}\text{e}^{-2x^2}\text{d}x$, and this requires to undo some of the above work in order to be able to use N(0, 1) and the statistics tables provided in the formula book.

$$\frac{1}{2} = \frac{2}{\sqrt{2\pi}}\int_0^{m} 2\text{e}^{-2x^2}\text{d}x = 2 \times \frac{1}{\sqrt{2\pi}}\int_0^{2m}\text{e}^{-\frac{1}{2}t^2}\text{d}t = 2\left\{\Phi(2m) - \tfrac{1}{2}\right\} \text{ or } \Phi\left(\tfrac{1}{2}m\right) = \frac{3}{4}$$

Use of the $N(0, 1)$ tables then gives $2m = 0.6745$ (0.675-ish) and m = 0.337 or 0.338 .

13 For A: p(launch fails) = p(>1 fail) = $1 - p_0 - p_1 = 1 - q^4 - 4q^3p$

so that E(repair) = Σx p(x) = $0.q^4 + K.4q^3p + 4K(1 - q^4 - 4q^3p)$

$$= 4K\left[q^3p + (1-q)(1+q+q^2+q^3) - 4q^3p\right]$$

$$= 4Kp\left[1+q+q^2-2q^3\right]$$

For B: p(launch fails) = p(>2 fail) = $1 - p_0 - p_1 - p_2 = 1 - q^6 - 6q^5p - 15q^4p^2$

so that E(repair) = Σx p(x)

$$= 0.q^6 + K.6q^5p + 2K.15q^4p^2 + 6K(1 - q^6 - 6q^5p - 15q^4p^2)$$

$$= 6K\left[q^5p + 5q^4p^2 + (1-q)(1+q+q^2+q^3+q^4+q^5) - 6q^5p - 15q^4p^2\right]$$

Extracting the p and obtaining the remaining in terms of q only,

$$= 6Kp\left[q^5 + 5q^4(1-q) + 1 + q + q^2 + q^3 + q^4 + q^5 - 6q^5 - 15q^4(1-q)\right]$$

$$= 6Kp\left[1+q+q^2+q^3-9q^4+6q^5\right]$$

Setting Rep(A) = $\frac{2}{3}$ Rep(B) $\Rightarrow$ $12Kp\left[1+q+q^2-2q^3\right] = 2Kp\left[1+q+q^2+q^3-9q^4+6q^5\right]$

Clearly, $p = 0$ is one solution and the rest simplifies to

$$0 = 3q^3(1 - 3q + 2q^2) = 3q^3(1-q)(1-2q).$$

We thus have $p = 1, 0, \frac{1}{2}$, with the 0 and 1 being rather trivial solutions.

STEP III, Solutions
2009

Section A: Pure Mathematics

1. The result for p can be found via calculating the equation of the line SV ($y - ms = \frac{ms - nv}{s - v}(x - s)$) or similar triangles. The result for q follows from that for p (given in the question) by suitable interchange of letters to give

$$q = \frac{(m-n)tu}{mt - nu}$$

As S and T lie on the circle, s and t are solutions of the equation

$\lambda^2 + (m\lambda - c)^2 = r^2$ i.e. $\left(1 + m^2\right)\lambda^2 - 2mc\lambda + \left(c^2 - r^2\right) = 0$

and so from considering sum and product of roots, $st = \frac{c^2 - r^2}{1 + m^2}$, and $s + t = \frac{2mc}{1 + m^2}$

Similarly $uv = \frac{c^2 - r^2}{1 + n^2}$, and $u + v = \frac{2nc}{1 + n^2}$ can be deduced by interchanging letters.

Substituting from the earlier results $p + q = \frac{(m-n)sv}{ms - nv} + \frac{(m-n)tu}{mt - nu}$ which can be simplified to $\frac{(m-n)}{(ms - nv)(mt - nu)}\left(stm(u + v) - nuv(s + t)\right)$

and then substituting the sum and product results yields the required result.

2 (i) The five required results are straightforward to write down, merely observing that initial terms in the summations are zero.

(ii) Substituting the series from (i) in the differential equation yields that

$-a_1 + 3a_3x^2 + \left(8a_4 + 4a_0\right)x^3 + ... = 0$, after having collected like terms.

Thus, comparing constants and x^2 coefficients $a_1 = 0$ and $a_3 = 0$

Comparing coefficients of x^{n-1}, for $n \geq 4$, $n(n-1)a_n - na_n + 4a_{n-4} = 0$ which gives the required result upon rearrangement.

With $a_0 = 1$, $a_2 = 0$, and as $a_1 = 0$, and $a_3 = 0$, we find $a_4 = \frac{-1}{2!}$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$, $a_8 = \frac{1}{4!}$, etc.

Thus $y = 1 - \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{4!}\left(x^2\right)^4 - \frac{1}{6!}\left(x^2\right)^6 + ... = \cos\left(x^2\right)$

With $a_0 = 0$, $a_2 = 1$, $y = \left(x^2\right) - \frac{1}{3!}\left(x^2\right)^3 + \frac{1}{5!}\left(x^2\right)^5 - \frac{1}{7!}\left(x^2\right)^7 + ... = \sin\left(x^2\right)$

3. (i) Substituting the power series and tidying up the algebra yields

$$f(t) = \frac{1}{\left(1 + \frac{t}{2!} + \ldots\ldots\right)} \quad \text{and so } \lim_{t \to 0} f(t) = 1 \ .$$

Similarly, $$f'(t) = \frac{(e^t - 1) - te^t}{(e^t - 1)^2} = \frac{-\frac{1}{2} - t\left(\frac{1}{2!} - \frac{1}{3!}\right) - \ldots\ldots}{\left(1 + \frac{t}{2!} + \ldots\ldots\right)^2} \quad \text{and so } \lim_{t \to 0} f'(t) = \frac{-1}{2}$$

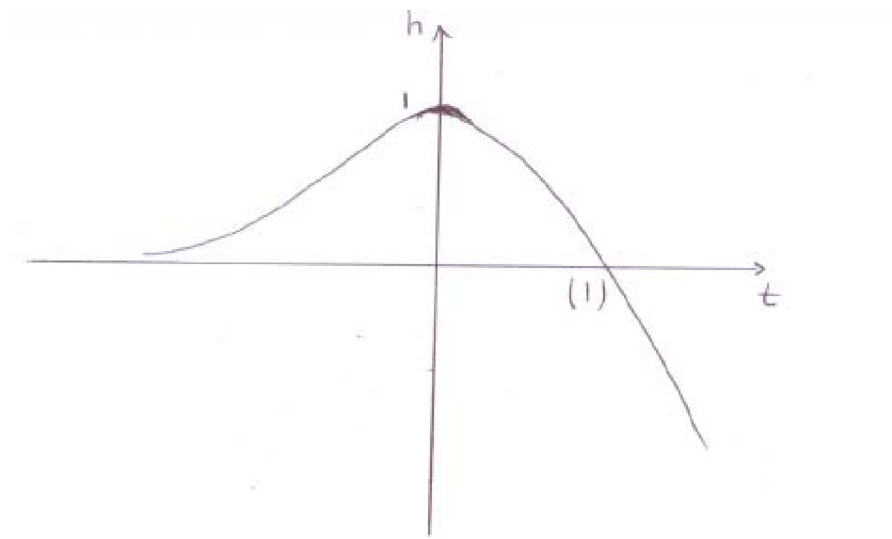
(Alternatively, this can be obtained by de l'Hopital.)

(ii) If we let $g(t) = f(t) + \frac{1}{2}t$, then simplifying the algebra gives

$$g(t) = \frac{t(e^t + 1)}{2(e^t - 1)}$$

after which it is can be shown by substituting –*t* for *t*
that $g(-t)$ is the same expression.

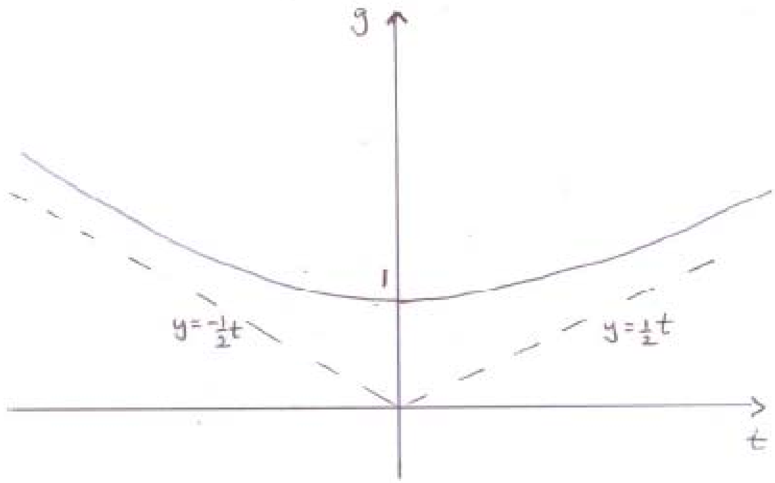
(iii) If we let $h(t) = e^t(1 - t)$, and find its stationary point, sketching the graph gives



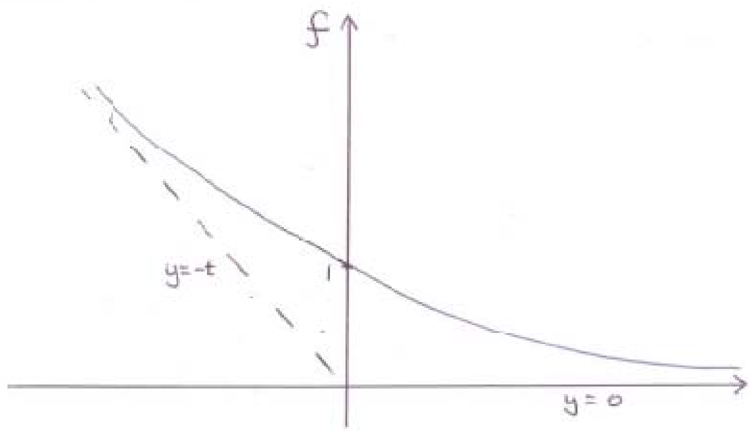
Hence $e^t(1 - t) \leq 1$ and so $e^t(1 - t) - 1 \leq 0$. (Alternatively, a sketch with e^t and $\frac{1}{1-t}$ will yield the result.)

Thus $f'(t) = \frac{(1-t)e^t - 1}{(e^t - 1)^2} \leq 0$, with equality only possible for $t = 0$, but we know $\lim_{t \to 0} f'(t) = \frac{-1}{2}$ and so, in fact, *f(t)* is always decreasing i.e. has no turning points.

Considering the graph of $g(t) = f(t) + \frac{1}{2}t$. It passes through (0,1), is symmetrical and approaches $y = \frac{1}{2}t$ as $t \to \infty$ and thus is



Therefore the graph of $f(t) = g(t) - \frac{1}{2}t$ also passes through (0,1), and has asymptotes $y = 0$ and $y = -t$ and thus is



4. (i) Substituting into the definition yields the Laplace transform as

$$\int_0^\infty e^{-st}e^{-bt}f(t)dt = \int_0^\infty e^{-t(s+b)}f(t)dt = F(s+b)$$

(ii) Similarly, a change of variable in the integral using $u = at$ yields the result.

(iii) Integrating by parts yields this answer.

(iv) A repeated integration by parts obtains

$$F(s) = 1 - s^2F(s)$$

which leads to the stated result.

Using the results obtained in the question, the transform of $\cos qt$ is

$q^{-1}\left(\dfrac{s/q}{s^2/q^2+1}\right)=\dfrac{s}{s^2+q^2}$,and so the transform of $e^{-pt}\cos qt$ is $\dfrac{(s+p)}{(s+p)^2+q^2}$

5. The first result may be obtained by considering

$$(x+y+z)^2-(x^2+y^2+z^2)=2(yz+zx+xy) \quad ,$$

the second by

$$(x^2+y^2+z^2)(x+y+z)=x^3+y^3+z^3+(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y)$$

and the third by

$$(x+y+z)^3=(x^3+y^3+z^3)+3(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y)+6xyz$$

Considering sums and products of roots, we can deduce that x satisfies the cubic equation $x^3-x^2-\frac{1}{2}x-\frac{1}{6}=0$, as do y and z by symmetry. Multiplying by x^{n-2} , $x^{n+1}=x^n+\frac{1}{2}x^{n-1}+\frac{1}{6}x^{n-2}$, with similar results for y and z. Summing these yields

$$S_{n+1}=S_n+\tfrac{1}{2}S_{n-1}+\tfrac{1}{6}S_{n-2}$$

Alternatively,

$x^{n+1}+y^{n+1}+z^{n+1}=(x+y+z)(x^n+y^n+z^n)-(xy^n+xz^n+yx^n+yz^n+zx^n+zy^n)$

$\quad =1.S_n-(xy+yz+zx)(x^{n-1}+y^{n-1}+z^{n-1})+xyz(x^{n-2}+y^{n-2}+z^{n-2})$

to give the result.

6. Using Euler, $e^{i\beta}-e^{i\alpha}=(\cos\beta-\cos\alpha)+i(\sin\beta-\sin\alpha)$

and so

$$\left|e^{i\beta}-e^{i\alpha}\right|^2=(\cos\beta-\cos\alpha)^2+(\sin\beta-\sin\alpha)^2$$

which can be expanded, and then using Pythagoras, compound and half angle formulae this becomes

$4\sin^2\dfrac{1}{2}(\beta-\alpha)$

$\left|e^{i\beta}-e^{i\alpha}\right|=2\sin\dfrac{1}{2}(\beta-\alpha)$ as both expressions are positive.

Alternative methods employ the factor formulae.

$\left|e^{i\alpha}-e^{i\beta}\right|\left|e^{i\gamma}-e^{i\delta}\right|+\left|e^{i\beta}-e^{i\gamma}\right|\left|e^{i\alpha}-e^{i\delta}\right|$

$=2\sin\left(\dfrac{1}{2}(\alpha-\beta)\right)2\sin\left(\dfrac{1}{2}(\gamma-\delta)\right)+2\sin\left(\dfrac{1}{2}(\beta-\gamma)\right)2\sin\left(\dfrac{1}{2}(\alpha-\delta)\right)$

which by use of the factor formulae and cancelling terms may be written

$2\left(\cos\left(\dfrac{1}{2}(\alpha-\beta-\gamma+\delta)\right)-\cos\left(\dfrac{1}{2}(\beta-\gamma+\alpha-\delta)\right)\right)$

and then again by factor formulae,

$2\sin\left(\frac{1}{2}(\alpha-\gamma)\right)2\sin\left(\frac{1}{2}(\beta-\delta)\right)$

which is

$\left|e^{i\alpha}-e^{i\gamma}\right|\left|e^{i\beta}-e^{i\delta}\right|$ as required.

Thus, the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite pairs of sides **(** Ptolemy's Theorem).

7. (i) This result is simply obtained using the principle of mathematical induction. The $n = 1$ case can be established merely by obtaining f_1 and f_2 from the definition, and then substituting these along with f_0.

(ii)

$$P_0(x) = \left(1+x^2\right)\frac{1}{1+x^2} = 1$$

$$P_1(x) = \left(1+x^2\right)^2\frac{-2x}{\left(1+x^2\right)^2} = -2x$$

$$P_2(x) = \left(1+x^2\right)^3\frac{6x^2-2}{\left(1+x^2\right)^3} = 6x^2-2$$

$$P_{n+1}(x) - \left(1+x^2\right)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x)$$

which differentiating P_n by the product rule and substituting

$$= \left(1+x^2\right)^{n+2} f_{n+1}(x) - \left(1+x^2\right)\left(\left(1+x^2\right)^{n+1} f_{n+1}(x) + (n+1)2x\left(1+x^2\right)^n f_n(x)\right) + 2(n+1)x\left(1+x^2\right)^{n+1} f_n(x)$$

which is zero.

Again using the principle of mathematical induction and the result just obtained, it can be found that $P_{k+1}(x)$ is a polynomial of degree <u>not greater than</u> $k+1$.

Further, assuming that $P_k(x)$ has term of highest degree, $(-1)^k (k+1)!\, x^k$, as $P_{n+1}(x) - \left(1+x^2\right)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x) = 0$, the term of highest degree of $P_{k+1}(x)$ is

$(-1)^k (k+1)!\, kx^{k-1}x^2 - 2(k+1)x(-1)^k (k+1)!\, x^k$

$= (-1)^{k+1}(k+2)!\, x^{k+1}$ as required.

(The form of the term need not be determined, but it must be shown to be non-zero.)

8. (i) Letting $x = e^{-t}$,

$\lim_{x\to 0}[x^m(\ln x)^n] = \lim_{t\to\infty}[(e^{-t})^m(-t)^n] = (-1)^n \lim_{x\to 0}[e^{-mt}t^n] = 0$

and so letting $m = n = 1$, $\lim_{x\to 0}[x\ln x] = 0$.

Thus, $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1$

(ii) Integrating by parts,

$$I_{n+1} = \int_0^1 x^m (\ln x)^{n+1} dx = \left[\frac{x^{m+1}(\ln x)^{n+1}}{m+1}\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\frac{(n+1)(\ln x)^n}{x} dx$$

$$= 0 - 0\ (using\ the\ first\ result) - \int_0^1 \frac{n+1}{m+1} x^m (\ln x)^n dx = -\frac{n+1}{m+1} I_n$$

So $I_n = \frac{-n}{m+1} \times \frac{-(n-1)}{m+1} \times \frac{-(n-2)}{m+1} \times \dots \times \frac{-1}{m+1} I_0 = \frac{(-1)^n n!}{(m+1)^n} \int_0^1 x^m dx$

$= \frac{(-1)^n n!}{(m+1)^{n+1}}$

(iii) $\int_0^1 x^x dx = \int_0^1 e^{x \ln x} dx = \int_0^1 1 + x \ln x + \frac{x^2 (\ln x)^2}{2!} + \dots\dots dx$

$= 1 + I_1 + \frac{1}{2!} I_2 + \dots\dots \quad = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \dots\dots$ as required.

Section B: Mechanics

9. (i) With V as the speed of projection from P, x and y the horizontal and vertical displacements from P at a time t after projection, and T the time of flight from P to Q, then

$x = Vt\cos\theta$, $y = Vt\sin\theta - \frac{1}{2}gt^2$, $\dot{x} = V\cos\theta$, and $\dot{y} = V\ sin\ \theta - gt$

So $\tan\alpha = \frac{VT\tan\theta - \frac{1}{2}gT^2}{VT\cos\theta} = \tan\theta - \frac{gT}{2V\cos\theta}$, and $\tan\varphi = \frac{V\sin\theta - gT}{V\cos\theta} = \tan\theta - \frac{gT}{V\cos\theta}$

Thus $\tan\theta + \tan\varphi = 2\tan\theta - \frac{gT}{V\cos\theta} = 2\tan\alpha$

(ii) Using the trajectory equation written as a quadratic equation in $\tan\theta$,

$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(\frac{gx^2}{2V^2} + y\right) = 0$, giving $\tan\theta + \tan\theta' = \frac{2V^2}{gx}$, and
$\tan\theta\tan\theta' = 1 + \frac{2V^2 y}{gx^2} = 1 + \frac{2V^2}{gx}\tan\alpha$.

Applying the compound angle formula and substituting, $\tan(\theta + \theta') = -\cot\alpha$
So, $+\theta' = \frac{\pi}{2} + \alpha + n\pi$, and as $0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $0 < \alpha < \frac{\pi}{2}$,
$\theta + \theta' = \frac{\pi}{2} + \alpha$.
Reversing the motion we have, $(-\varphi) + (-\varphi') = \frac{\pi}{2} + (-\alpha) + n'\pi$, and therefore,
$\varphi + \varphi' = \alpha + \left(-n' - \frac{1}{2}\right)\pi = \theta + \theta' - n''\pi$
$0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi' < \frac{\pi}{2}$, and $\varphi < \theta$, $\varphi' < \theta'$
so $\varphi + \varphi' = \theta + \theta' - \pi$, or as required $\theta + \theta' = \varphi + \varphi' + \pi$

10. Supposing that the particle P has mass m, the spring has natural length l, and modulus of elasticity λ, $mg = \frac{\lambda d}{l}$

If the speed of P when it hits the top of the spring is v, then $v = \sqrt{2gh}$

By Newton's second law, the second-order differential equation is thus

$m\ddot{x} = mg - \frac{\lambda x}{l} = mg - \frac{mgx}{d}$ and so $\ddot{x} = g - \frac{gx}{d}$ with initial conditions that $x = 0$, $\dot{x} = \sqrt{2gh}$, when $t = 0$.

$\ddot{x} + \frac{gx}{d} = g$ has complementary function $x = B\cos\omega t + C\sin\omega t$

where $\omega = \sqrt{\frac{g}{d}}$, and particular integral $x = A$, where $A = d$.

The initial conditions yield, $B = -d$ and $C = \sqrt{2dh}$

So $x = d - d\cos\sqrt{\frac{g}{d}}t + \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$.

$d\cos\sqrt{\frac{g}{d}}t - \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$ may be expressed in the form $R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$ where

$R^2 = d^2 + 2dh$, and $\tan\alpha = \frac{\sqrt{2dh}}{d} = \sqrt{\frac{2h}{d}}$

So $x = d - R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$

$x = 0$ next when $t = T$, that is when $2\pi - \left(\sqrt{\frac{g}{d}}T + \alpha\right) = \alpha$

So $\sqrt{\frac{g}{d}}T = 2\pi - 2\alpha = 2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}$ and $T = \sqrt{\frac{d}{g}}\left(2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}\right)$.

11. (i) Conserving momentum yields $MV = M(1 + bx)v$ and so $V = (1 + bx)v$ Written as $V = (1 + bx)\frac{dx}{dt}$, separating variables and integrating

$Vt + c = x + \frac{1}{2}bx^2$, but as $= 0$, when $t = 0$, $c = 0$

So $\frac{1}{2}bx^2 + x - Vt = 0$, and so $x = \frac{-1\pm\sqrt{1+2bVt}}{b}$, except $x > 0$, and thus

$x = \frac{-1+\sqrt{1+2bVt}}{b}$

(ii) $Mf = \frac{d}{dt}(mv) = \frac{d}{dt}(M(1 + bx)v)$

So, $ft + c' = (1 + bx)v$ and as $= 0$, $x = 0$, and $v = V$ we have $c' = V$.

Thus $v = \frac{ft+V}{1+bx}$ as required.

Separating variables and integrating

$\frac{1}{2}ft^2 + Vt + c'' = x + \frac{1}{2}bx^2$ and as $x = 0$, when $t = 0$, $c'' = 0$

So $\frac{1}{2}bx^2 + x - \frac{1}{2}ft^2 - Vt = 0$, and so $= \frac{-1\pm\sqrt{1+fbt^2+2bVt}}{b}$, except $x > 0$, and thus

$x = \frac{-1+\sqrt{1+fbt^2+2bVt}}{b}$

If $1 + fbt^2 + 2bVt$ is a perfect square, then x will be linear in t and $\frac{dx}{dt}$ will be constant, i.e. if $4b^2V^2 - 4fb = 0$, that is $bV^2 = f$
(in which case $x = \frac{-1+\sqrt{1+b^2V^2t^2+2bVt}}{b} = \frac{-1+(1+bVt)}{b} = Vt$, and $v = V$ as expected.)

Otherwise, $= \frac{ft+V}{1+bx} = \frac{ft+V}{\sqrt{1+fbt^2+2bVt}} = \frac{f+\frac{V}{t}}{\sqrt{fb+\frac{2bV}{t}+\frac{1}{t^2}}}$, and as $t \to \infty$, $v \to \frac{f}{\sqrt{fb}} = \sqrt{\frac{f}{b}}$, a constant, as required.

Section C: Probability and Statistics

12. (i) $E(X_1) = \frac{1}{2}k$, $E(X_2|X_1 = x_1) = \frac{1}{2}x_1$, and so
$E(X_2) = \sum \frac{1}{2}x_1 \, P(X_1 = x_1) = \frac{1}{2}E(X_1) = \frac{1}{4}k$
$\sum_{i=1}^{\infty} E(X_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i k = k$ using the sum of an infinite GP.

(ii) $G_Y(t) = E(t^Y) = E\left(t^{\sum_{i=1}^{k} Y_i}\right) = \prod_{i=1}^{k} E(t^{Y_i})$
$P(Y_i = 0) = \frac{1}{2}$, $(Y_i = 1) = \frac{1}{4}$, … , $P(Y_i = r) = \left(\frac{1}{2}\right)^{r-1}$
and so $E(t^{Y_i}) = \frac{1}{2} + \frac{1}{4}t + \cdots + \left(\frac{1}{2}\right)^{r-1} t^r + \ldots = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}t\right)} = \frac{1}{2-t}$ (infinite GP)
Thus $G_Y(t) = \prod_{i=1}^{k} \frac{1}{2-t} = \left(\frac{1}{2-t}\right)^k$

$G'_Y(t) = \frac{k}{(2-t)^{k+1}}$, $G''_Y(t) = \frac{k(k+1)}{(2-t)^{k+2}}$, and $G^{(r)}{}_Y(t) = \frac{k(k+1)(k+2)\ldots(k+r-1)}{(2-t)^{k+r}}$
and so $E(Y) = G'_Y(1) = k$, $Var(Y) = G''_Y(1) + G'_Y(1) - \left(G'_Y(1)\right)^2 = 2k$
and $P(Y = r) = \frac{G^{(r)}{}_Y(0)}{r!} = \frac{k(k+1)(k+2)\ldots(k+r-1)}{2^{k+r}r!} = {}^{k+r-1}C_r \left(\frac{1}{2}\right)^{k+r}$ for $r = 0, 1, 2, \ldots$

(Alternatively, $P(Y = r)$ is coefficient of t^r in $G_Y(t)$ which can be expanded binomially to yield the same result.)

13.
(i) $F(x) = P(X < x) = P(\cos\theta < x) = P(\cos^{-1} x < \theta < 2\pi - \cos^{-1} x)$
Therefore, $F(x) = \frac{2\pi - 2\cos^{-1} x}{2\pi}$
So $(x) = \frac{dF}{dx} = \frac{1}{\pi\sqrt{1-x^2}}$, for $-1 \le x \le 1$

$E(X) = 0$
$E(X^2) = \int_{-1}^{1} x^2 \frac{1}{\pi\sqrt{1-x^2}} \quad dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 u}{\pi} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2u}{2\pi} \, du = \frac{1}{2}$
So $Var(X) = \frac{1}{2}$
If $X = x$, $Y = \pm\sqrt{1-x^2}$ equiprobably, so $E(XY) = 0$, $E(Y) = 0$ and thus $Cov(X,Y) = 0$, and hence $Corr(X,Y) = 0$.

X and Y are not independent for if $X = x$, $Y = \pm\sqrt{1 - x^2}$ only, whereas without the restriction, Y can take all values in $[-1,1]$.

(ii) $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = 0$, and $E(\bar{Y}) = 0$ similarly.

$E(\bar{X}\bar{Y}) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i \sum_{j=i}^{n} Y_j\right) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i\, Y_i\right)$ as X_i, Y_j are independent and each have expectation zero.

$E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i\, Y_i\right) = 0$ from part (i), and so $E(\bar{X}\bar{Y}) = 0$. Thus $Cov(\bar{X},\bar{Y}) = 0$, and hence $Corr(\bar{X},\bar{Y}) = 0$ as required.

For large n , $\bar{X} \sim N\left(0, \frac{1}{2n}\right)$ approximately, by Central Limit Theorem.

Thus,

$$P\left(|\bar{X}| \leq \sqrt{\frac{2}{n}}\right) \approx P\left(|z| \leq \frac{\sqrt{\frac{2}{n}}}{\frac{1}{\sqrt{2n}}}\right) = P(|z| \leq 2) \approx P(|z| \leq 1 \cdot 960) \approx 0 \cdot 95$$