STEP 2008, Paper 3, Q8 - Solution (2 pages; 5/6/18)

(i) $p = -\frac{a_{r+1}}{a_r} = -\frac{1}{2}$

$$\left(1 - \frac{1}{2}x\right)S = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \cdots$$

$$-\frac{1}{6}x^2 - \frac{1}{12}x^3 - \cdots \quad = \frac{1}{3}x$$

so that $S = \frac{\frac{1}{3}x}{1-\frac{1}{2}x} = \frac{2x}{6-3x}$

Let S' be the sum of the first $n + 1$ terms of S.

Then

$$\left(1 - \frac{1}{2}x\right)S' = \frac{1}{3}x + \frac{1}{6}x^2 + \cdots + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} x^n + \frac{1}{3}\left(\frac{1}{2}\right)^n x^{n+1}$$

$$-\frac{1}{6}x^2 - \frac{1}{12}x^3 - \cdots - \frac{1}{3}\left(\frac{1}{2}\right)^n x^{n+1} - \frac{1}{3}\left(\frac{1}{2}\right)^{n+1} x^{n+2}$$

$$= \frac{1}{3}x - \frac{1}{3}\left(\frac{1}{2}\right)^{n+1} x^{n+2}$$

so that $S' = \frac{2x - \left(\frac{1}{2}\right)^n x^{n+2}}{6-3x}$

[which is consistent with the expression for S]

[The official sol'ns invoke the sum of a GP for this last part, despite the fact that we weren't allowed to do that for the infinite series.]

(ii) Substituting in the recurrence relation, to find p & q:

$18 + 8p + 2q = 0 \Rightarrow 9 + 4p + q = 0 \Rightarrow q = -(4p + 9)$

& $37 + 18p + 8q = 0$

so that $37 + 18p - 8(4p + 9) = 0 \Rightarrow 14p = -35 \Rightarrow p = -\frac{5}{2}$

and $q = 1$

Consider $(1 + px + qx^2)T$

$= 2 + 8x + 18x^2 + 37x^3 + \cdots$

$+px(2 + 8x + 18x^2 + \cdots)$

$+qx^2(2 + 8x + \cdots)$

$= 2 + (8 + 2p)x + x^2(18 + 8p + 2q) + x^3(37 + 18p + 8q) + \cdots$

$= 2 + 3x$, as subsequent coefficients are zero, from the recurrence relation

[Note the advantage of keeping the p & q in until the last moment: had we substituted their values, but made an arithmetic mistake somewhere, then the recurrence relation might have been hidden.]

Thus $T = \frac{2+3x}{1-\frac{5x}{2}+x^2} = \frac{2(2+3x)}{2x^2-5x+2} = \frac{2(2+3x)}{(2x-1)(x-2)}$

To find the partial fractions,

$T = \frac{A}{2x-1} + \frac{B}{x-2}$,

where $A(x-2) + B(2x-1) = 4 + 6x$

Then $x = 2 \Rightarrow 3B = 16 \Rightarrow B = \frac{16}{3}$

and $x = \frac{1}{2} \Rightarrow -\frac{3A}{2} = 7 \Rightarrow A = -\frac{14}{3}$,

so that $T = \frac{14}{3}(1-2x)^{-1} - \frac{8}{3}\left(1 - \frac{x}{2}\right)^{-1}$

Replacing the sums to infinity of these GPs with the sums of the first $n + 1$ terms then gives

$\frac{14\left(1-(2x)^{n+1}\right)}{3(1-2x)} - \frac{8\left(1-\left(\frac{x}{2}\right)^{n+1}\right)}{3(1-\frac{x}{2})}$