STEP 2008, Paper 3, Q5 - Solution (2 pages; 5/6/18)

rtp (result to prove): $[T_n(x)]^2 - T_{n-1}(x)T_{n+1}(x) = f(x) \quad for\ n \geq 1$ (1),

given that $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$ (2)

When $n = 1, LHS = [T_1(x)]^2 - T_0(x)T_2(x) = f(x)$, so that (1) is true for $n = 1$.

Assume that (1) is true for $n = k$.

[Because of the complexity of the expressions involved, we should be wary of getting bogged down in algebra.]

In order to establish (1) for $n = k + 1$, we want to show that

$[T_{k+1}(x)]^2 - T_k(x)T_{k+2}(x) = [T_k(x)]^2 - T_{k-1}(x)T_{k+1}(x)$ (3)

[This equation involves $T_{k-1}(x), T_k(x), T_{k+1}(x)$ & $T_{k+2}(x)$; (2) could be used to remove the $T_{k+2}(x)$, and make some progress. However, after factorising (3) a simpler approach is revealed. When dealing with algebra, factorising is generally the first option to be explored, whilst expanding brackets is generally the last. Also, rearranging (3) so that one side is zero is likely to be helpful.]

(3) $\Rightarrow T_{k+1}(x)[T_{k+1}(x) + T_{k-1}(x)] - T_k(x)[T_{k+2}(x) + T_k(x)] = 0$
(4) is the result to be proved

Then, from (2), LHS of (4) $= T_{k+1}(x)[2xT_k(x)] - T_k(x)[2xT_{k+1}(x)] = 0$, as required.

Thus we have established (1) for $n = k + 1$, and shown that if (1) is true for $n = k$, then it is true for $n = k + 1$.

As (1) is true for $n = 1$, it follows that it is true for $n = 2,3, \dots$ and hence for all integer $n \geq 1$, by the principle of induction.

When $f(x) = 0$, we have:

$[T_n(x)]^2 - T_{n-1}(x)T_{n+1}(x) = 0$ (5)

and $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$ (2)

[As we are trying to find an expression for $T_n(x)$ in terms of $T_0(x)$, we want some sort of 'reduction formula', giving $T_n(x)$ in terms of $T_{n-1}(x)$, and possibly $T_{n-2}(x)$.]

Eliminating $T_{n+1}(x)$ from (5) & (2) gives:

$[T_n(x)]^2 - T_{n-1}(x)[2xT_n(x) - T_{n-1}(x)\,] = 0$

So we have a quadratic for $T_{n-1}(x)$, allowing $T_{n-1}(x)$ to be expressed in terms of $T_n(x)$. As we would prefer it the other way round, we could instead eliminate $T_{n-1}(x)$ from (5) & (2), to give:

$[T_n(x)]^2 - [2xT_n(x) - T_{n+1}(x)]T_{n+1}(x) = 0$

Denote $T_n(x)$ by T_n

Then ${T_{n+1}}^2 - 2xT_nT_{n+1} + {T_n}^2 = 0$

$\Rightarrow T_{n+1} = \frac{2xT_n \pm \sqrt{4x^2{T_n}^2 - 4{T_n}^2}}{2} = T_n(x \pm \sqrt{x^2 - 1})$

So $T_n(x) = T_0(x)[r(x)]^n$, where $r(x) = x \pm \sqrt{x^2 - 1}$