STEP 2008, Paper 3, Q2 - Solution (2 pages; 14/4/21)

(i) **1st part**

Consider $\sum_{r=0}^{n}[(r+1)^k - r^k] = (n+1)^k$

LHS $= \sum_{r=0}^{n}[1 + kr + \binom{k}{2}r^2 + \cdots + \binom{k}{k-1}r^{k-1}$

$= (n+1) + kS_1(n) + \binom{k}{2}S_2(n) + \cdots + \binom{k}{k-1}S_{k-1}(n)$

$= (n+1) + \binom{k}{k-1}S_1(n) + \binom{k}{k-2}S_2(n)$

$+ \cdots + \binom{k}{3}S_{k-3}(n) + \binom{k}{2}S_{k-2}(n) + kS_{k-1}(n)$, so that

$kS_{k-1}(n) = (n+1)^k - (n+1) - \binom{k}{2}S_{k-2}(n) - \binom{k}{3}S_{k-3}(n)$

$- \cdots - \binom{k}{k-1}S_1(n)$, as required.

2nd part

When $k = 4$ this gives

$4S_3(n) = (n+1)^4 - (n+1) - 6S_2(n) - 4S_1(n)$, so that

$S_3(n) = \frac{1}{4}\{(n+1)^4 - (n+1) - n(n+1)(2n+1) - 2n(n+1)\}$

$= \frac{1}{4}(n+1)\{(n^3 + 3n^2 + 3n + 1) - 1 - (2n^2 + n) - 2n\}$

$= \frac{1}{4}(n+1)(n^3 + n^2)$

$= \frac{1}{4}n^2(n+1)^2$

3rd part

When $k = 5$ (*) gives

$5S_4(n) = (n+1)^5 - (n+1) - 10S_3(n) - 10S_2(n) - 5S_1(n)$, so that $S_4(n) = \frac{1}{5}\{(n+1)^5 - (n+1) - \frac{5}{2}n^2(n+1)^2$

$-\frac{5}{3}n(n+1)(2n+1) - \frac{5}{2}n(n+1)\}$

$= \frac{(n+1)}{30}\{6(n+1)^4 - 6 - 15n^2(n+1) - 10n(2n+1) - 15n\}$ (A)

Now, $(n+1)^4 - 1 = [(n+1)^2 + 1][(n+1)^2 - 1]$

$= [(n+1)^2 + 1].n(n+2)$,

so that (A) $= \frac{n(n+1)}{30}\{6(n^2+2n+2)(n+2) - 15n(n+1)$

$-10(2n+1) - 15\}$

$= \frac{n(n+1)}{30}\{6n^3 + n^2(12+12-15) + n(24+12-15-20)$

$+(24-10-15)\}$

$= \frac{n(n+1)}{30}\{6n^3 + 9n^2 + n - 1\}$

This can be simplified to $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$

[though it would be time-consuming to consider all possible factors of the form $an+1$ (where a might be negative)]

(ii) **1st part**

From (*) it can be seen that if $S_{k-2}(n)$ is of order $k-1$, then (due to the term $(n+1)^k$) $S_{k-1}(n)$ will be of order k. As $S_1(n)$ is of order 2, it follows by induction that $S_k(n)$ is of order $k+1$, for all $k \geq 1$.

2nd part

The constant term is $S_k(0) = \sum_{r=0}^{0} r^k = 0$

3rd part

The sum of the coefficients is $S_k(1) = \sum_{r=0}^{1} r^k = 1$