STEP 2008, Paper 3, Q1 - Solution (2 pages; 13/5/20)

$a + b = 1$ (1)

$ax + by = \frac{1}{3}$ (2)

$ax^2 + by^2 = \frac{1}{5}$ (3)

$ax^3 + by^3 = \frac{1}{7}$ (4)

$(2) \Rightarrow (ax + by)(x + y) = \frac{1}{3}(x + y)$

$\Rightarrow ax^2 + axy + bxy + by^2 = \frac{1}{3}(x + y)$

$\Rightarrow \frac{1}{5} + xy = \frac{1}{3}(x + y)$, from (3) & (1)

or $\frac{1}{5} + q = \frac{p}{3}$, writing $p = x + y$ & $q = xy$

or $3 + 15q = 5p$ (5)

Also, $(3) \Rightarrow (ax^2 + by^2)(x + y) = \frac{1}{5}(x + y)$

$\Rightarrow ax^3 + ax^2y + by^2x + by^3 = \frac{1}{5}(x + y)$

$\Rightarrow \frac{1}{7} + xy(ax + by) = \frac{1}{5}(x + y)$, from (4)

$\Rightarrow \frac{1}{7} + q\left(\frac{1}{3}\right) = \frac{p}{5}$, from (2)

or $15 + 35q = 21p$ (6)

Then, $7(5) - 3(6) \Rightarrow 21 - 45 = (35 - 63)p$

so that $p = \frac{-24}{-28} = \frac{6}{7}$, and from (5):

$$q = \frac{1}{15}\left(\frac{30}{7} - 3\right) = \frac{1}{15(7)}(30 - 21) = \frac{3}{35}$$

So $x + y = \frac{6}{7}$ and $xy = \frac{3}{35}$

and hence x & y are the roots of $z^2 - \frac{6}{7}z + \frac{3}{35} = 0$

or $35z^2 - 30z + 3 = 0$

$\Rightarrow x, y$ are $\frac{30 \pm \sqrt{900 - 420}}{70} = \frac{30 \pm 4\sqrt{30}}{70} = \frac{3}{7} \pm \frac{2\sqrt{30}}{35}$ or $\frac{3}{7} \pm \frac{2\sqrt{6}}{7\sqrt{5}}$

If $x = \frac{3}{7} + \frac{2\sqrt{6}}{7\sqrt{5}}, y = \frac{3}{7} - \frac{2\sqrt{6}}{7\sqrt{5}}$

Then $(2) \Rightarrow a\left(\frac{3}{7} + \frac{2\sqrt{30}}{35}\right) + b\left(\frac{3}{7} - \frac{2\sqrt{30}}{35}\right) = \frac{1}{3}$

and $a + b = 1$ from (1),

so that $a\left(\frac{3}{7} + \frac{2\sqrt{30}}{35}\right) + (1 - a)\left(\frac{3}{7} - \frac{2\sqrt{30}}{35}\right) = \frac{1}{3}$

$\Rightarrow a\left(\frac{4\sqrt{30}}{35}\right) = \frac{1}{3} - \frac{3}{7} + \frac{2\sqrt{30}}{35}$

$\Rightarrow a(12\sqrt{30}) = 35 - 45 + 6\sqrt{30}$

$\Rightarrow a = \frac{6\sqrt{30} - 10}{12\sqrt{30}} = \frac{6(30) - 10\sqrt{30}}{12(30)} = \frac{18 - \sqrt{30}}{36} = \frac{1}{2} - \frac{\sqrt{5}}{6\sqrt{6}}$

and $b = 1 - a = \frac{1}{2} + \frac{\sqrt{5}}{6\sqrt{6}}$

If $x = \frac{3}{7} - \frac{2\sqrt{6}}{7\sqrt{5}}, y = \frac{3}{7} + \frac{2\sqrt{6}}{7\sqrt{5}}$:

$a = \frac{1}{2} + \frac{\sqrt{5}}{6\sqrt{6}}$ & $b = \frac{1}{2} - \frac{\sqrt{5}}{6\sqrt{6}}$