**STEP 2008, Paper 3, Q13 - Solution** (2 pages; 5/6/18)

E(number of rings created at the 1st step)

= P(same strings)× 1+P(different strings)× 0

=P(2nd end is from the same string as the 1st end)

$= \frac{1}{2n-1}$

After the 1st step, either a ring has been created, leaving $n-1$ pieces of string, or no ring has been created, but again leaving $n-1$ pieces of string. So, after the 1st step, the process starts again, with $n-1$ pieces of string.

Hence E(number of rings created by the end of the process)

$= \frac{1}{2n-1} + \frac{1}{2(n-1)-1} + \cdots$

$= \frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{1}$

(when there is one string left, a ring has to be formed)

$= \sum_{r=1}^{n} \frac{1}{2r-1}$ [reversing the order of the series]

Let $Y = X_1 + \cdots + X_n$ be the total number of rings created, where $X_r$ is the number of rings created by selecting 2 ends from r strings (so that $X_r = 0\ or\ 1$), so that $P(X_r = 1) = \frac{1}{2r-1}$

Then $Var(Y) = \sum_{r=1}^{n} Var(X_r)$

and $Var(X_r) = E\left({X_r}^2\right) - (E(X_r))^2$

$= \left\{\frac{1}{2r-1}(1^2) + 0\right\} - \left(\frac{1}{2r-1}\right)^2$

$= \frac{(2r-1)-1}{(2r-1)^2} = \frac{2(r-1)}{(2r-1)^2}$

Thus $Var(Y) = 2\sum_{r=1}^{n}\frac{r-1}{(2r-1)^2} = 2\sum_{r=2}^{n}\frac{r-1}{(2r-1)^2}$

E(number of rings created when $n = 40$) $= \sum_{r=1}^{40000}\frac{1}{2r-1}$

$= (\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{79999})$

$= \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{80000}\right) - (\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{80000})$

$\approx ln80000 - \frac{1}{2}ln40000$

$= \frac{1}{2}ln\left(\frac{80000^2}{40000}\right) = \frac{1}{2}ln160000 = ln400 = 2ln20 \approx 2(3) = 6$

[Note that Probability (and also Mechanics) questions quite often involve parts that are nothing to do with the subject!]