STEP 2008, Paper 2, Q1 – Solution (2 pages; 2/6/18)

(i) In order for the 2nd term to equal the 1st term,

${x_1}^2 - {y_1}^2 + 1 = x_1$ (A) & $2x_1y_1 + 1 = y_1$ (B)

(B) $\Rightarrow y_1(1 - 2x_1) = 1$

Then (A) $\Rightarrow {x_1}^2 - x_1 + 1 = \frac{1}{(1-2x_1)^2}$

$\Rightarrow ({x_1}^2 - x_1 + 1)(1 - 4x_1 + 4{x_1}^2) = 1$

$\Rightarrow {x_1}^2 - 4{x_1}^3 + 4{x_1}^4 - x_1 + 4{x_1}^2 - 4{x_1}^3 + 1 - 4x_1 + 4{x_1}^2 = 1$

$\Rightarrow x_1(4{x_1}^3 - 8{x_1}^2 + 9x_1 - 5) = 0$

As $x_1 = 1$ is a root of $4{x_1}^3 - 8{x_1}^2 + 9x_1 - 5 = 0$, this can be written as $x_1(x_1 - 1)(4{x_1}^2 - 4x_1 + 5) = 0$

Then, as the quadratic has no real roots, $x_1 = 0 \text{ } or \text{ } 1$

and, from (B), the required values of (x_1, y_1) are $(0, 1) \text{ } and \text{ } (1, -1)$, as the 3rd term will equal the 2nd term etc, by repeating the process exactly as before, so that the sequence is constant.

(ii) If $(x_1, y_1) = (-1,1)$, $(x_2, y_2) = (a, b)$

and $(x_3, y_3) = (a^2 - b^2 + a, 2ab + b + 2)$

A necessary condition for the period to be 2 is that

$(x_3, y_3) = (x_1, y_1)$,

so that $a^2 - b^2 + a = -1$ (C) & $2ab + b + 2 = 1$ (D)

$(D) \Rightarrow b(2a + 1) = -1$

Then $(C) \Rightarrow a^2 + a + 1 = \frac{1}{(2a+1)^2}$

$\Rightarrow (a^2 + a + 1)(4a^2 + 4a + 1) = 1$

$\Rightarrow 4a^4 + 4a^3 + a^2 + 4a^3 + 4a^2 + a + 4a^2 + 4a + 1 = 1$

$\Rightarrow a(4a^3 + 8a^2 + 9a + 5) = 0$

As $a = -1$ is a root of $4a^3 + 8a^2 + 9a + 5 = 0$,

this can be written as $a(a + 1)(4a^2 + 4a + 5) = 0$

Then, as the quadratic has no real roots, $a = 0\ or - 1$

If $a = 0, b = -1$, from (D), and if $a = -1, b = 1$

In the latter case, the 1st 3 terms are equal, so that the sequence doesn't have period 3.

When $a = 0$ & $b = -1$, the 1st two terms are different. Also, the 4th term will be (a, b), as the 3rd term equals the 1st term, which produced (a, b) as the next term; and so on.

Thus $a = 0$ & $b = -1$.