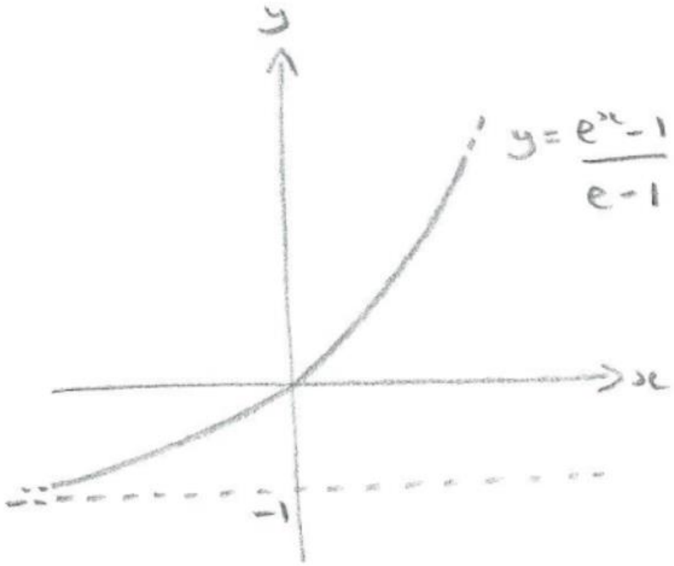
STEP 2008, Paper 1, Q6 - Solution (4 pages; 26/4/21)

1st part

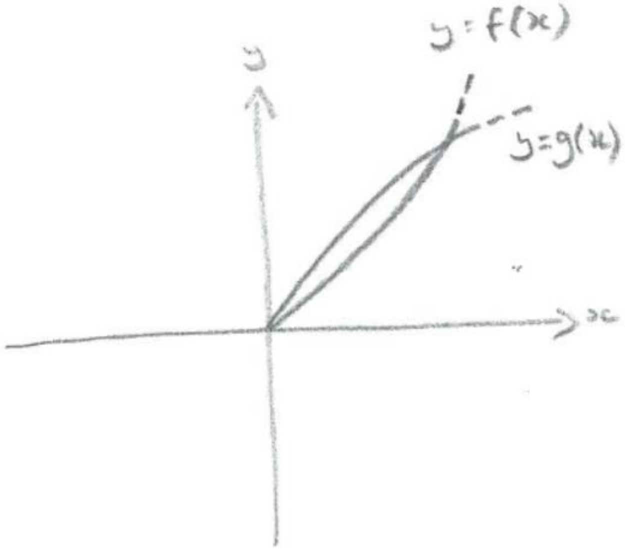
The graph of $y = \frac{e^x - 1}{e - 1}$ can be obtained from the graph of $y = e^x$ by a translation of $\binom{0}{-1}$, followed by a stretch in the y direction of scale factor $\frac{1}{e-1}$ (which is less than 1).



To investigate the slope of the curve, relative to $y = x$:

$f(x) = \frac{e^x - 1}{e - 1} \Rightarrow f'(x) = \frac{e^x}{e-1}$, and $f'(0) = \frac{1}{e-1} < 1$

Limiting the domain to $x \geq 0$ then gives $y = f(x)$, and $y = g(x)$ is the reflection of $y = f(x)$ in the line $y = x$.



2nd part

Let $I_1 = \int_0^{\frac{1}{2}} \frac{e^x - 1}{e-1}\ dx = \frac{1}{e-1}[e^x - x]_0^{\frac{1}{2}}$

$= \frac{1}{e-1}\left(e^{\frac{1}{2}} - \frac{1}{2}\right) - \frac{1}{e-1}(1) = \frac{e^{\frac{1}{2}} - \frac{3}{2}}{e-1}$

Now $y = \frac{e^x - 1}{e-1} \Rightarrow (e-1)y = e^x - 1$

$\Rightarrow x = \ln\{(e-1)y + 1\}$, and so $g(x) = \ln\{(e-1)x + 1\}$

Let $I_2 = \int_0^k \ln\{(e-1)x + 1\}\, dx$

$=$ [by Parts] $\left[x \ln\{(e-1)x + 1\}{}_0^k\right] - \int_0^k \frac{x(e-1)}{(e-1)x+1} dx$

$= (k ln\{(e-1)k + 1\} - 0) - \int_0^k 1 - \frac{1}{(e-1)x+1}\ dx$

$= \frac{1}{\sqrt{e}+1} ln\{(\sqrt{e} - 1) + 1\} - [x - ln\{(e-1)x + 1\}.\frac{1}{e-1}]_0^k$

$= \frac{1}{\sqrt{e}+1} \cdot \frac{1}{2} - \left(k - ln\{(e-1)k + 1\}.\frac{1}{e-1}\right) + (0)$

$= \frac{1}{2(\sqrt{e}+1)} - \frac{1}{(\sqrt{e}+1)} + \frac{1}{2(e-1)}$

$= -\frac{1}{2(\sqrt{e}+1)} + \frac{1}{2(e-1)}$

And then $I_1 + I_2 = \frac{e^{\frac{1}{2}}-\frac{3}{2}}{e-1} - \frac{1}{2(\sqrt{e}+1)} + \frac{1}{2(e-1)}$

$= \frac{1}{2(e-1)}\{2e^{\frac{1}{2}} - 3 - (\sqrt{e} - 1) + 1\}$

$= \frac{1}{2(e-1)}(\sqrt{e} - 1) = \frac{1}{2(\sqrt{e}+1)}$, as required.

3rd part

Note now that $f\left(\frac{1}{2}\right) = \frac{\sqrt{e}-1}{e-1} = \frac{1}{\sqrt{e}+1} = k$

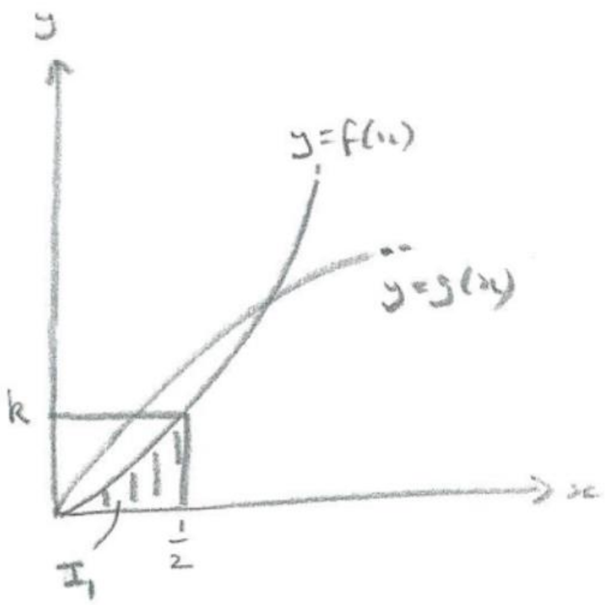
and, as g is the inverse of f, $g(k) = \frac{1}{2}$

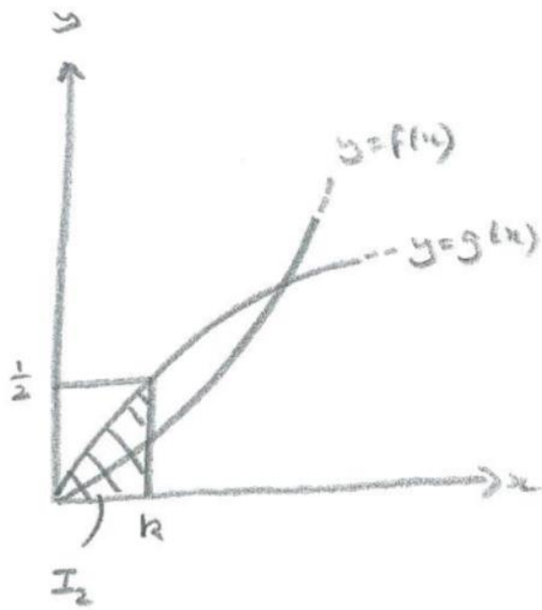
Also, $k = \frac{1}{\sqrt{e}+1} < \frac{1}{1+1} = \frac{1}{2}$

I_1 and I_2 are the shaded areas in the diagrams below [as $k < \frac{1}{2}$, the rectangle shown in the left-hand diagram appears to the left of the intersection of f & g, where $f(x) < x$] , and by the symmetry between f and g ($g(x)$ bears the same relation to the y-axis as $f(x)$ does to the x-axis),

$I_2 =$ area of rectangle with base $\frac{1}{2}$ and height k (in the left-hand diagram) $-I_1$,

and so $I_1 + I_2 = \frac{1}{2}k = \frac{1}{2(\sqrt{e}+1)}$, as required.


y
y=f(x)
y=g(x)
k
x
1/2
I1


y
y=f(x)
y=g(x)
1/2
x
k
I2