STEP 2008, Paper 1, Q2 - Solution (2 pages; 1/6/2018)

$\frac{dt}{dx} = 1 + ½\ (x^2 + 2bx + c)^{-1/2}$(2x+2b)

$= 1 + \frac{x+b}{t-x} = \frac{t-x+x+b}{t-x} = \frac{t+b}{t-x}$

provided t-x $\neq 0$: the discriminant of $x^2 + 2bx + c$ is $(2b)^2$ – 4c

$= 4(b^2$-c)<0; hence $x^2 + 2bx + c \neq 0$, and t-x = $\sqrt{x^2 + 2bx + c}$ $\neq 0$

[The condition b^2<c should, in theory, prompt the idea of " b^2-4ac" (with different use of letters obviously)]

Hence $\frac{dx}{dt} = \frac{t-x}{t+b}$

$\int \frac{dx}{\sqrt{x^2+2bx+c}} = \int \frac{dx}{t-x} = \int \frac{dt}{t+b} =$ ln|t+b| + C

$=$ ln|x+$\sqrt{x^2 + 2bx + c}$ + b| + C (A)

[Although the moduli signs are often glossed over at A Level (and questions are chosen so that the quantity is positive), this is a typical STEP refinement that needs to be allowed for.]

As b^2<c, $x^2 + 2bx + c > (x + b)^2$,

and so x+$\sqrt{x^2 + 2bx + c}$ + b > 0 [noting that $\sqrt{x}$ always means the positive root]

and the integral can be written as ln(x+$\sqrt{x^2 + 2bx + c}$ + b) + C

If b^2= c, then $\int \frac{dx}{\sqrt{x^2+2bx+c}} = \int \frac{dx}{|x+b|}$ (B)

If x+b>0, then (A) = ln 2(x+b) + C = ln2 + ln(x+b) + C

= ln|x+b| + C^1

and (B) = $\int \frac{dx}{x+b}$ = ln(x+b) + C, which agrees with (A)

If x+b<0, then (A) = ln 0 + C, which is undefined

whereas (B) = - $\int \frac{dx}{x+b}$ = - ln|x+b| + C = -ln(-(x+b)) + C

Thus (A) does not hold if x+b<0.