STEP 2008, Paper 1, Q1 (3 pages, 13/2/24)

1st Part

x is irrational if it cannot be written in the form $\frac{p}{q}$, where p & $q \in \mathbb{Z}$ $(q \neq 0)$

2nd Part

A: Suppose that both of p and q are rational.

Then $pq = \frac{a}{b} . \frac{c}{d} = \frac{ac}{bd}$, where a, b, c & $d \in \mathbb{Z}$, contradicting the fact that pq is irrational. Hence at least one of p and q is irrational.

B: Similarly, if both of p and q are rational, then $p + q = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, contradicting the fact that $p + q$ is irrational. Hence at least one of p and q is irrational.

3rd Part

C: Let $p = \pi$ & $q = -\pi$. Then $p + q = 0$, which is rational.

4th Part

To show that no pair exists for which both numbers are rational:

Case (i) Suppose that $p = \pi + e$ & $q = \pi - e$ are both rational.

Then $p + q = 2\pi$, which is irrational, since if it were rational, $2\pi = \frac{a}{b}$ and $\pi = \frac{a}{2b}$, contradicting the fact that π is irrational.

[It might be a bit over-the-top to prove that 2π is irrational, but the nature of the question suggests that it could be required. In fact the official solutions don't bother with this.]

But B then implies that at least one of p & q is irrational, contradicting our supposition.

Case (ii) Suppose that $p = \pi + e$ & $q = \pi^2 - e^2$ are both rational. Then $\frac{q}{p}$ is rational [hopefully this doesn't need to be proved], and thus

$\frac{\pi^2-e^2}{\pi+e} = \pi - e$ is rational. But this gives case (i), which leads to a contradiction.

Case (iii) Suppose that $p = \pi + e$ & $q = \pi^2 + e^2$ are both rational. Then

$p^2 - q = 2\pi e$, and hence πe must be rational, contradicting the fact that $e\pi$ is irrational.

Case (iv) Suppose that $p = \pi - e$ & $q = \pi^2 - e^2$ are both rational. Then

$\frac{q}{p}$ is rational, and hence $\frac{\pi^2-e^2}{\pi-e} = \pi + e$ is rational. But this gives case (i), which leads to a contradiction.

Case (v) Suppose that $p = \pi - e$ & $q = \pi^2 + e^2$ are both rational. Then

$q - p^2 = 2\pi e$, and hence πe must be rational, contradicting the fact that $e\pi$ is irrational.

Case (vi) Suppose that $p = \pi^2 - e^2$ & $q = \pi^2 + e^2$ are both rational. Then $p + q = 2\pi^2$, which is irrational (since π^2 is assumed to be irrational). But B then implies that at least one of p & q is irrational, contradicting our supposition.

Thus two pairs of rational numbers cannot be found amongst the 4 given numbers, so that at most one of them is rational.

[A slight cause for concern here is that we have not made use of the given facts that e & e^2 are irrational. The official sol'ns use the same method though.]