STEP 2007, Paper 3, Q3 – Solution (2 pages; 31/5/18)

(i) $2, 3, 5, 8, 13, 21$

(ii) $F_{2k+3}F_{2k+1} - F^2{}_{2k+2} = (F_{2k+1} + F_{2k+2})F_{2k+1} - F^2{}_{2k+2}$

$= F^2{}_{2k+1} + F_{2k+2}(F_{2k+1} - F_{2k+2})$

$= F^2{}_{2k+1} - F_{2k+2}(F_{2k+2} - F_{2k+1})$

$= -F_{2k+2}F_{2k} + F^2{}_{2k+1}$, as required

(iii) For $n = 1$: LHS $= F_3F_1 - {F_2}^2 = (2)(1) - 1^2 = 1$;

so the result is true for $n = 1$

Now assume that the result is true for $n = k$,

so that $F_{2k+1}F_{2k-1} - {F_{2k}}^2 = 1$ (A)

We want to show that $F_{2(k+1)+1}F_{2(k+1)-1} - {F_{2(k+1)}}^2 = 1$ (B)

ie that $F_{2k+3}F_{2k+1} - {F_{2k+2}}^2 = 1$

By (ii), LHS $= -F_{2k+2}F_{2k} + F^2{}_{2k+1}$

[we now need to involve (A) somehow]

$= -(F_{2k} + F_{2k+1})F_{2k} + F^2{}_{2k+1}$

$= -{F_{2k}}^2 + F_{2k+1}(-F_{2k} + F_{2k+1})$

$= -{F_{2k}}^2 + F_{2k+1}F_{2k-1} = 1$, by (A)

So we have shown that, if the result is true for $n = k$, then it is true for $n = k + 1$. Then, as it is true for $n = 1$, it is therefore true for $n = 2, 3$, … and all positive integer n, by the principle of induction.

As $F_{2n+1}F_{2n-1} - {F_{2n}}^2 = 1$, ${F_{2n}}^2 + 1 = F_{2n+1}F_{2n-1}$ is divisible by F_{2n+1}

(iv) From (iii), $F_{2n+1}F_{2n-1} - {F_{2n}}^2 = 1$

$\Rightarrow F_{2n+1}F_{2n-1} - (F_{2n+1} - F_{2n-1})^2 = 1$

$\Rightarrow F_{2n+1}F_{2n-1} - {F_{2n+1}}^2 - {F_{2n-1}}^2 + 2F_{2n+1}F_{2n-1} = 1$

$\Rightarrow {F_{2n-1}}^2 + 1 = F_{2n+1}(3F_{2n-1} - F_{2n+1})$,

and is thus divisible by F_{2n+1}