STEP 2007, Paper 3, Q2 – Solution (2 pages; 31/5/18)

(i) $1.3.5.7 \ldots (2n-1) = \frac{(2n)!}{2.4.6\ldots(2n)} = \frac{(2n)!}{2^n n!}$

For $|-4x| < 1$ and hence $|x| < \frac{1}{4}$,

$(1-4x)^{-\frac{1}{2}}$

$= 1 + \left(-\frac{1}{2}\right)(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)(-4x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)(-4x)^3}{2!} + \cdots$

$= 1 + (1)(2x) + \frac{(1)(3)(2x)^2}{2!} + \frac{(1)(3)(5)(2x)^3}{3!} + \cdots$

$= 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!} \frac{(2x)^n}{n!}$, from (i)

$= 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$, as required

(ii) Differentiating,

$\left(-\frac{1}{2}\right)(1-4x)^{-\frac{3}{2}}(-4) = \sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2} x^{n-1}$

Then let $x = \frac{6}{25}$ ($<\frac{6}{24} = \frac{1}{4}$), so that

$2\left(1 - \frac{24}{25}\right)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n-1)!} \left(\frac{6}{25}\right)^{n-1}$

and hence $2(125) = \left(\frac{25}{6}\right) \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n-1)!} \left(\frac{6}{25}\right)^n$,

so that $\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n-1)!} \left(\frac{6}{25}\right)^n = 60$, as required

(iii) [The presence of n+1 in the denominator suggests that integration may be required.]

Integrating (i),

$$\frac{(1-4x)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\left(-\frac{1}{4}\right) = x + \{\sum_{n=1}^{\infty}\frac{(2n)!}{(n+1)(n!)^2}x^{n+1}\} + c$$

$$x = 0 \Rightarrow -\frac{1}{2} = c$$

[Looking at the quantities in the 'show that' result that are being raised to the power of n:]

Let $x = \frac{2}{3^2}\left(< \frac{2}{8} = \frac{1}{4}\right)$,

so that $-\left(\frac{1}{2}\right)\left(1-\frac{8}{9}\right)^{\frac{1}{2}} = \frac{2}{9} + \{\sum_{n=1}^{\infty}\frac{(2n)!}{(n+1)(n!)^2}\left(\frac{2}{3^2}\right)^{n+1}\} - \frac{1}{2}$

and hence $-\frac{1}{6} - \frac{2}{9} + \frac{1}{2} = \sum_{n=1}^{\infty}\frac{2^{n+1}(2n)!}{3^{2n+2}(n+1)!n!}$

giving $\frac{-3-4+9}{18} = \frac{1}{9}\sum_{n=1}^{\infty}\frac{2^{n+1}(2n)!}{3^{2n}(n+1)!n!}$

and hence $\sum_{n=1}^{\infty}\frac{2^{n+1}(2n)!}{3^{2n}(n+1)!n!} = 1$, as required