STEP 2007, Paper 3, Q12 – Sol'n (3 pages; 20 /4/21)

1st part

$E(N) = \sum_{r=1}^{2n-1} r \, . \, \frac{1}{2n-1} = \frac{1}{2n-1} \, . \frac{1}{2}(2n-1)(2n) = n$

2nd part

$E(N^2) = \sum_{r=1}^{2n-1} r^2 \, . \, \frac{1}{2n-1} = \frac{1}{2n-1} \, . \frac{1}{6}(2n-1)(2n)(2[2n-1]+1)$

$= \frac{1}{3}n(4n-1)$

3rd part

$E(Y) = \sum_{r=1}^{2n-1} P(N=r)E(Y|N=r)$

$= \sum_{r=1}^{2n-1} \frac{1}{2n-1} E(\sum_{k=1}^{r} X_k) = \frac{1}{2n-1}\sum_{r=1}^{2n-1} r\mu = \frac{\mu}{2n-1} . \frac{1}{2} \ (2n-1)(2n)$

$= n\mu$, as required

4th part

$\text{Cov}(Y, N) = E(YN) - E(Y)E(N)$

Now, $E(YN) = \sum_{r=1}^{2n-1} P(N=r)E(YN|N=r)$

$= \sum_{r=1}^{2n-1} \frac{1}{2n-1} rE(\sum_{k=1}^{r} X_k) = \frac{1}{2n-1}\sum_{r=1}^{2n-1} r(r\mu)$

$= \frac{\mu}{2n-1} \, . \frac{1}{6}(2n-1)(2n)(2[2n-1]+1) = \frac{\mu n(4n-1)}{3}$

So $\text{Cov}(Y, N) = \frac{\mu n(4n-1)}{3} - n\mu . n = \frac{\mu n}{3}(4n-1-3n) = \frac{1}{3}n(n-1)\mu$

as required

5th part

$Var(Y) = E(Y^2) - [E(Y)]^2$

and $E(Y^2) = \sum_{r=1}^{2n-1} P(N=r) \, E(Y^2|N=r)$

$= \frac{1}{2n-1}\sum_{r=1}^{2n-1} E(X_1 + \cdots + X_r)^2$

Now, $E(X_1+\cdots+X_r)^2 = rE\left({X_1}^2\right) + 2\binom{r}{2}E(X_1X_2)$

$= rE\left({X_1}^2\right) + \frac{2r(r-1)}{2!}E(X_1)E(X_2)$,

as the X_i have the same mean and variance (and therefore $E({X_i}^2)$ and are independent (so that $E\left(X_iX_j\right) = E(X_i)E(X_j)$)

$= rE\left({X_1}^2\right) + r(r-1)\mu^2$

Now, $\sigma^2 = E\left({X_1}^2\right) - \mu^2$,

so that $E(X_1+\cdots+X_r)^2 = r(\sigma^2+\mu^2) + r(r-1)\mu^2$

And so $E(Y^2) = \frac{1}{2n-1}\sum_{r=1}^{2n-1}\{r(\sigma^2+\mu^2)+r(r-1)\mu^2\}$

$= \frac{1}{2n-1}\sum_{r=1}^{2n-1}\{r\sigma^2 + r^2\mu^2\}$

$= \frac{\sigma^2}{2n-1}\,.\frac{1}{2}(2n-1)(2n) + \frac{\mu^2}{2n-1}\,.\frac{1}{6}(2n-1)(2n)(4n-2+1)$

$= n\sigma^2 + \frac{\mu^2 n(4n-1)}{3}$

Hence $Var(Y) = \left\{n\sigma^2 + \frac{\mu^2n(4n-1)}{3}\right\} - (n\mu)^2$

$= n\sigma^2 + \frac{\mu^2 n(4n-1-3n)}{3} = n\sigma^2 + \frac{\mu^2n(n-1)}{3}$

[A standard result (which probably couldn't just be quoted for this question) is that $Var(Y) = E_N(Var(Y|N)) + Var_N(E(Y|N))$,

where N is randomly chosen from the integers 1 to n.

(See Prob. & Stats page: "Laws of Total Expectation & Variance".)

Now, $Var(Y|N) = Var(X_1+X_2+\cdots+X_N) = N\sigma^2$,

and so $E_N(Var(Y|N) = E_N(N\sigma^2) = \sigma^2.\frac{1}{2}(n+1)$

And $Var_N\big(E(Y|N)\big) = Var_N(N\mu) = \mu^2 Var_N(N)$,

and $Var_N(N) = E(N^2) - [E(N)]^2$

$= \{\sum_{r=1}^{n} r^2 \,.\frac{1}{n}\} - [\frac{1}{2}(n+1)]^2$

$= \frac{1}{n}\,.\frac{1}{6}n(n+1)(2n+1) - \frac{1}{4}(n+1)^2$

$= \frac{(n+1)}{12}[2(2n+1) - 3(n+1)] = \frac{(n+1)(n-1)}{12}$

So $Var(Y) = \frac{\sigma^2}{2}(n+1) + \frac{\mu^2(n+1)(n-1)}{12}$

Then, replacing n with $2n-1$,

$Var(Y) = \frac{\sigma^2}{2}(2n) + \frac{\mu^2(2n)(2n-2)}{12} = \sigma^2 n + \frac{\mu^2 n(n-1)}{3}$]