STEP 2007, Paper 2, Q5 – Solution (2 pages; 15/4/21)

(i) **1st part**

$$f^2(x) = \frac{\left(\frac{x+\sqrt{3}}{1-\sqrt{3}x}\right)+\sqrt{3}}{1-\sqrt{3}\left(\frac{x+\sqrt{3}}{1-\sqrt{3}x}\right)} = \frac{x+\sqrt{3}+\sqrt{3}-3x}{1-\sqrt{3}x-\sqrt{3}x-3} = \frac{2\sqrt{3}-2x}{-2\sqrt{3}x-2} = \frac{x-\sqrt{3}}{\sqrt{3}x+1}$$

2nd part

$$f^3(x) = \frac{\left(\frac{x-\sqrt{3}}{1+\sqrt{3}x}\right)+\sqrt{3}}{1-\sqrt{3}\left(\frac{x-\sqrt{3}}{1-\sqrt{3}x}\right)} = \frac{x-\sqrt{3}+\sqrt{3}+3x}{1+\sqrt{3}x-\sqrt{3}x+3} = \frac{4x}{4} = x$$

3rd part

If p is a positive integer, and $q = 0, 1\ or\ 2$,

$f^{3p+q}(x) = f^q f^3 \dots f^3\left(f^3(x)\right) = f^q f^3 \dots f^3(x) = \cdots = f^q(x)$;

ie $f^n(x)$ has period 3 wrt n

And so $f^{2007}(x) = x$, as 2007 is a multiple of 3

(ii) $f(tan\theta) = \frac{tan\theta+\sqrt{3}}{1-\sqrt{3}tan\theta} = \tan\left(\theta + \frac{\pi}{3}\right)$,

so that $f^n(x) = \tan\left(\theta + \frac{n\pi}{3}\right)$ when $n = 1$

From (i), $f^2(x) = \frac{tan\theta-\sqrt{3}}{\sqrt{3}tan\theta+1} = \tan\left(\theta - \frac{\pi}{3}\right) = \tan\left(\theta - \frac{\pi}{3} + \pi\right)$

$= \tan\left(\theta + \frac{2\pi}{3}\right)$

so that $f^n(x) = \tan\left(\theta + \frac{n\pi}{3}\right)$ when $n = 2$

From (i), $f^3(x) = x = tan\theta = \tan(\theta + \pi)$,

so that $f^n(x) = \tan\left(\theta + \frac{n\pi}{3}\right)$ when $n = 3$

Also, $\tan\left(\theta + \frac{n\pi}{3} + k\pi\right) = \tan\,(\theta + \frac{(n+3k)\pi}{3})$,

so that $\tan\,(\theta + \frac{n\pi}{3})$ also has period 3 wrt n,

and thus $f^n(x) = \tan\,(\theta + \frac{n\pi}{3})$ for all positive integer n, as required.

(iii) If $t = cos\theta$, then $g(t) = \cos\,(\theta - \frac{\pi}{6})$ for $0 \leq \theta \leq \pi$ (so that $sin\theta = \sqrt{1-t^2} \geq 0$).

[Alternatively, we could set $t = sin\theta$, with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so that $cos\theta = \sqrt{1-t^2} \geq 0$, and then $g(t) = \sin\,(\theta + \frac{\pi}{6})$.]

Then $g^2(t) = g\left(\cos\left(\theta - \frac{\pi}{6}\right)\right) = \cos\,(\left[\theta - \frac{\pi}{6}\right] - \frac{\pi}{6})$

and so $g^n(t) = \cos(\theta - \frac{n\pi}{6})$ where $t = cos\theta$ and $0 \leq \theta \leq \pi$;

ie $\theta = arccost$

[or $g^n(t) = \sin\left(\theta + \frac{n\pi}{6}\right)$, where $t = sin\theta$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$;

ie $\theta = arcsint$]