STEP 2007, Paper 2, Q4 – Solution (2 pages; 23/5/18)

The equation can be written as

$\alpha sinAcosB - \alpha cosAsinB + \beta cosAcosB - \beta sinAsinB$

$-\gamma sinAcosB - \gamma cosAsinB = 0$

dividing by $cosAcosB$ (as $cosA$ & $cosB$ are non-zero)

$\Leftrightarrow \alpha tanA - \alpha tanB + \beta - \beta tanAtanB - \gamma tanA - \gamma tanB = 0$

$\Leftrightarrow tanAtanB - \frac{(\alpha-\gamma)tanA}{\beta} + \frac{(\alpha+\gamma)tanB}{\beta} - 1 = 0$

which, with $m = \frac{-(\alpha+\gamma)}{\beta}$ & $n = \frac{\alpha-\gamma}{\beta}$

$\Leftrightarrow (tanA - m)(tanB - n) = 0$ when $\alpha^2 - \gamma^2 = \beta^2$ (1)

ie when $\alpha^2 = \beta^2 + \gamma^2$

[The official solution shows that, for "if and only if" proofs, it may be sufficient to indicate that the line of reasoning is reversible (assuming that this is the case).]

(i) With $\alpha = 2, \beta = \sqrt{3}$ & $\gamma = 1, A = x$ & $B = \frac{\pi}{4}, \beta^2 + \gamma^2 = \alpha^2$,

so that the equation $\Leftrightarrow \left(tanx + \sqrt{3}\right)\left(tan\left(\frac{\pi}{4}\right) - \frac{1}{\sqrt{3}}\right) = 0$, from (1)

$\Rightarrow tanx = -\sqrt{3}$

$\Rightarrow x = -\frac{\pi}{3} + \pi \ or \ -\frac{\pi}{3} + 2\pi$

ie $x = \frac{2\pi}{3} \ or \ \frac{5\pi}{3}$

(ii) With $\alpha = 2, \beta = \sqrt{3}$ & $\gamma = 1, A = x$ & $B = \frac{\pi}{6}, \beta^2 + \gamma^2 = \alpha^2$, so that the equation $\Leftrightarrow \left(tanx + \sqrt{3}\right)\left(tan\left(\frac{\pi}{6}\right) - \frac{1}{\sqrt{3}}\right) = 0$

As $tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, the original equation is true for all values of x

(iii) [Forcing the LHS into the earlier form:]

Let $A = \frac{1}{2}\left(\left[x + \frac{\pi}{3}\right] + 3x\right) = 2x + \frac{\pi}{6}$,

so that $B = 3x - \left(2x + \frac{\pi}{6}\right) = x - \frac{\pi}{6}$

Then with $\alpha = 2, \beta = \sqrt{3}$ & $\gamma = 1$ again,

$\left(\tan(2x + \frac{\pi}{6}) + \sqrt{3}\right)\left(tan\left(x - \frac{\pi}{6}\right) - \frac{1}{\sqrt{3}}\right) = 0$ (2)

$0 \leq x < 2\pi \Rightarrow \frac{\pi}{6} \leq 2x + \frac{\pi}{6} < 4\pi + \frac{\pi}{6}$

and $-\frac{\pi}{6} \leq x - \frac{\pi}{6} < 2\pi - \frac{\pi}{6}$

Then (2) $\Rightarrow 2x + \frac{\pi}{6} = -\frac{\pi}{3} + \pi$ or $-\frac{\pi}{3} + 2\pi$ or $-\frac{\pi}{3} + 3\pi$

or $-\frac{\pi}{3} + 4\pi$

or $x - \frac{\pi}{6} = \frac{\pi}{6}$ or $\frac{\pi}{6} + \pi$

ie $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{3}$ or $\frac{4\pi}{3}$