STEP 2007, Paper 2, Q3 – Solution (2 pages; 23/5/18)

$x = atan\theta \Rightarrow dx = asec^2\theta\, d\theta$

and hence $\int \frac{1}{a^2+x^2}\, dx = \int \frac{asec^2\theta}{a^2+a^2tan^2\theta} d\theta = \frac{1}{a}\int d\theta$

$= \frac{\theta}{a} + c = \frac{1}{a} arctan\left(\frac{x}{a}\right) + c$

(i)(a) $I = [arctan(sinx)]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

(b) $t = \tan\left(\frac{x}{2}\right) \Rightarrow dt = \frac{1}{2}sec^2\left(\frac{x}{2}\right)dx = \frac{1}{2}(t^2+1)\, dx$

Also $tanx = \frac{2t}{1-t^2}$, and the hypotenuse of the right-angled triangle with other sides of $2t$ & $1-t^2$ is $\sqrt{4t^2+(1-t^2)^2} = 1+t^2$

so that $cosx = \frac{1-t^2}{1+t^2}$ & $sinx = \frac{2t}{1+t^2}$

Then $\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(1-t^2)(1+t^2)}{(1+t^2)^2+4t^2} dx$

$\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(1-t^2)/(1+t^2)}{1+[\frac{4t^2}{(1+t^2)^2}]} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{cosx}{1+sin^2x} dx = \frac{1}{2}I$, as required

(ii) If we make the same substitution as in (i)(b), then we obtain

$\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(1-t^2)(1+t^2)}{(1+t^2)^2+12t^2} dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(1-t^2)/(1+t^2)}{1+3[\frac{4t^2}{(1+t^2)^2}]} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{cosx}{1+3sin^2x} dx$

$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \frac{cosx}{\frac{1}{3}+sin^2x} dx = \frac{1}{6}\left[\left(\frac{1}{\left(\frac{1}{\sqrt{3}}\right)}\right) arctan\left(\frac{sinx}{\sqrt{\frac{1}{3}}}\right)\right]_0^{\frac{\pi}{2}}$

$$= \frac{\sqrt{3}}{6}\left(\frac{\pi}{3} - 0\right) = \frac{\pi\sqrt{3}}{18}$$