STEP 2007, Paper 1, Q7 – Solution (3 pages; 21/5/18)

(i) L_1 is $\underline{r} = \begin{pmatrix} 1+2\lambda \\ 2\lambda \\ 2-3\lambda \end{pmatrix}$ & L_2 is $\underline{r} = \begin{pmatrix} 4+\mu \\ 2\mu-2 \\ 9-2\mu \end{pmatrix}$

$D^2 = (1+2\lambda-[4+\mu])^2 + (2\lambda-[2\mu-2])^2 + (2-3\lambda-[9-2\mu])^2$

$= (-3+2\lambda-\mu)^2 + 4(\lambda-\mu+1)^2 + (-7-3\lambda+2\mu)^2$

$= (9+4+49) + \lambda^2(4+4+9) + \mu^2(1+4+4)$

$+2\lambda(-6+4+21) + 2\mu(3-4-14) + 2\lambda\mu(-2-4-6)$

$= 62 + 17\lambda^2 + 9\mu^2 + 38\lambda - 30\mu - 24\lambda\mu$

$= (\lambda-1)^2 + 36 + [25 + 40\lambda + 16\lambda^2 + 9\mu^2 - 30\mu - 24\lambda\mu\ ]$

and then $(3\mu-4\lambda-5)^2 = 9\mu^2 + 16\lambda^2 + 25 - 24\lambda\mu - 30\mu + 40\lambda$,

which equals the expression in the square brackets, so that the required result follows

ie $D^2 = (3\mu-4\lambda-5)^2 + (\lambda-1)^2 + 36$

D^2 is minimised when the two squared terms are both zero, giving

$D = \sqrt{36} = 6$

At this point, $\lambda = 1$ & $3\mu - 4(1) - 5 = 0$, so that $\mu = 3$.

Then the two coordinates are $\begin{pmatrix} 1+2(1) \\ 2(1) \\ 2-3(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

& $\begin{pmatrix} 4+3 \\ 2(3)-2 \\ 9-2(3) \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$

[strictly speaking, these should be written as $(3,2,-1)$& $(7,4,3)$]

(ii) As before, we obtain

$D^2 = (2-3-4\beta k)^2 + (3+\alpha-3-\beta+\beta k)^2 + (5+2+3\beta k)^2$

$= (1+4\beta k)^2 + (\alpha-\beta+\beta k)^2 + (7+3\beta k)^2$

[The question now is whether we need to expand this and look for a suitable arrangement into two squares plus a constant number, or is there a shortcut? The former method has several drawbacks: (a) it is time-consuming, (b) we might not find a suitable arrangement, and (c) we don't know whether more than one rearrangement is possible: if there is only one, it seems quite possible that we might not discover it! For these reasons, we can probably reject the latter approach as too risky, and concentrate on the former. The official solution uses the risky approach, but without explaining where the inspiration comes from when deriving the squared terms, beyond saying - rather unconvincingly - (in the examiners' report) that "the coefficients do not permit many possibilities". The approach adopted below is much simpler.]

The reason why, in (i), $D^2 = (3\mu-4\lambda-5)^2 + (\lambda-1)^2 + 36$ is an advantageous form is that only one squared term involves both λ & μ.

Note that in (ii) only the middle squared term involves both α & β (whereas each of the squared terms at the corresponding stage in (i) involve both λ & μ). So there is no need to expand everything: we can just expand the 1st and 3rd squared terms, and complete the square.

Thus $(1+4\beta k)^2 + (7+3\beta k)^2 = \beta^2(16k^2+9k^2) + \beta(8k + 42k) + 50$

$= 25k^2\beta^2 + 50k\beta + 50 = (5k\beta + 5)^2 + 25$

and $\quad D^2 = (\alpha - \beta + \beta k)^2 + (5k\beta + 5)^2 + 25$,

giving a minimum distance of $\sqrt{25} = 5$, provided that $k \neq 0$.
When $k = 0$, the minimum distance is $\sqrt{50}$.

When $k = 0$, both L_3 & L_4 have a direction vector of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; ie they are both parallel to the y-axis. When $k \neq 0$, the two lines are skew (ie not parallel and not intersecting).