STEP 2007, Paper 1, Q3 – Solution (3 pages; 21/5/18)

$cos^4\theta - sin^4\theta = (cos^2\theta - sin^2\theta)(cos^2\theta + sin^2\theta) = cos2\theta(1) = cos2\theta$,

as required

For $cos^4\theta + sin^4\theta$, consider

$1 = (cos^2\theta + sin^2\theta)^2 = cos^4\theta + sin^4\theta + 2cos^2\theta sin^2\theta$

Then $cos^4\theta + sin^4\theta = 1 - 2cos^2\theta sin^2\theta = 1 - \frac{1}{2}(2cos\theta sin\theta)^2$

$= 1 - \frac{1}{2}sin^2(2\theta)$, as required (A)

For the 1st integral, $cos^4\theta = \frac{1}{2}([cos^4\theta - sin^4\theta] + [cos^4\theta + sin^4\theta])$

$= \frac{1}{2}\cos(2\theta) + \frac{1}{2} - \frac{1}{4}sin^2(2\theta)$

We need to be able to integrate $sin^2(2\theta)$. We can write

$sin^2(2\theta) = \frac{1}{2}(1 - \cos(4\theta))$

So $\int_0^{\frac{\pi}{2}} cos^4\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos(2\theta) + \frac{1}{2} - \frac{1}{8}(1 - \cos(4\theta))d\theta$

$= [\frac{1}{4}\sin(2\theta) + \frac{3\theta}{8} + \frac{1}{32}\sin(4\theta)]_0^{\frac{\pi}{2}} = \frac{3\pi}{16}$

For the 2nd integral, $sin^4\theta = \frac{1}{2} - \frac{1}{4}sin^2(2\theta) - \frac{1}{2}\cos(2\theta)$

$= \frac{1}{2} - \frac{1}{8}(1 - \cos(4\theta)) - \frac{1}{2}\cos(2\theta)$

and hence $\int_0^{\frac{\pi}{2}} sin^4\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{8} + \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta)\, d\theta$

$[\frac{3\theta}{8}+\frac{1}{32}\sin(4\theta)-\frac{1}{4}\sin(2\theta)]_0^{\frac{\pi}{2}} = \frac{3\pi}{16}$

Thus $\int_0^{\frac{\pi}{2}} sin^4\theta\ d\theta = \int_0^{\frac{\pi}{2}} cos^4\theta\ d\theta$

This can be argued from the areas under the sine and cosine curves. But you can never be sure how convinced the examiners need to be. The following algebraic equivalent is probably safer:

Let $\phi = \frac{\pi}{2} - \theta$. Then $\int_0^{\frac{\pi}{2}} cos^4\theta\ d\theta = \int_0^{\frac{\pi}{2}} sin^4(\frac{\pi}{2} - \theta)\ d\theta = \int_{\frac{\pi}{2}}^0 sin^4\phi\ (-1)\ d\phi$

$= \int_0^{\frac{\pi}{2}} sin^4\phi\ d\phi$

$cos^6\theta - sin^6\theta = (cos^2\theta - sin^2\theta)(cos^4\theta + cos^2\theta sin^2\theta + sin^4\theta)$

$= \cos(2\theta)(1 - \frac{1}{2}sin^2(2\theta) + \frac{1}{4}sin^2(2\theta))$, using (A)

$= \cos(2\theta)(1 - \frac{1}{4}sin^2(2\theta))$ (B)

whilst $cos^6\theta + sin^6\theta = (cos^2\theta + sin^2\theta)(cos^4\theta - cos^2\theta sin^2\theta + sin^4\theta)$

$= (1)(1 - \frac{1}{2}sin^2(2\theta) - \frac{1}{4}sin^2(2\theta))$, using (A)

$= 1 - \frac{3}{4}sin^2(2\theta)$ (C)

Then $cos^6\theta = \frac{1}{2}\{\cos(2\theta)\left(1 - \frac{1}{4}sin^2(2\theta)\right) + 1 - \frac{3}{4}sin^2(2\theta)\}$, combining (B) & (C)

and hence $\int_0^{\frac{\pi}{2}} cos^6\theta\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\ d\theta -$
$\frac{1}{8}\int_0^{\frac{\pi}{2}} \cos(2\theta) sin^2(2\theta)\ d\theta$

$+\frac{\pi}{4} - \frac{3}{8}\int_0^{\frac{\pi}{2}} sin^2(2\theta)\ d\theta$

$= \frac{1}{2}\left[\frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}} - \frac{1}{8}\left[\frac{1}{2}\left(\frac{sin^3(2\theta)}{3}\right)\right]_0^{\frac{\pi}{2}} + \frac{\pi}{4} - \frac{3}{8}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(2\theta))d\theta$

$= 0 - 0 + \frac{\pi}{4} - \frac{3}{16}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4} - \frac{3}{16}\left(\frac{\pi}{2}\right) = \frac{5\pi}{32}$

and $\int_0^{\frac{\pi}{2}} sin^6\theta\ d\theta = \int_0^{\frac{\pi}{2}} cos^6\theta\ d\theta$, by the same method as before

Alternative approach

Reduction formulae can be obtained for
$\int_0^{\frac{\pi}{2}} sin^n\theta\ d\theta$ & $\int_0^{\frac{\pi}{2}} cos^n\theta\ d\theta$, giving $I_n = \frac{n-1}{n} I_{n-2}$ in both cases.

Thus, $\int_0^{\frac{\pi}{2}} cos^6\theta\ d\theta = \frac{5}{6}\int_0^{\frac{\pi}{2}} cos^4\theta\ d\theta = \frac{5}{6}\left(\frac{3\pi}{16}\right) = \frac{5\pi}{32}$

Induction can also be applied to the Reduction Formulae to show that, if $\int_0^{\frac{\pi}{2}} sin^{2k}\theta\ d\theta = \int_0^{\frac{\pi}{2}} cos^{2k}\theta\ d\theta$, then

$\int_0^{\frac{\pi}{2}} sin^{2(k+1)}\theta\ d\theta = \int_0^{\frac{\pi}{2}} cos^{2(k+1)}\theta\ d\theta$