STEP 2007, Paper 1, Q1 – Solution (4 pages; 21/5/18)

Introduction

Not the easiest Q1 on a STEP 1 paper! You could be forgiven for expecting there to be a clever way of answering part (i).

The approach to part (i) in the official sol'ns leads directly to the required generalisation in (ii), but this is not true of some other approaches, which then leave you stranded!

The moral here is to read the 2nd part of the question before answering the 1st part: the method used for the 1st part needs to be capable of generalisation.

A useful technique to be noted from this question is that of conditioning on something. The official solution does this on the sum of the 1st two digits, which gives 8 cases. The point is that the number of cases needs to be reasonably small. As an alternative, we could start to condition on the 1st digit. This seems more promising, as there are only 4 cases; though for each case we end up considering the possible 2nd digit.

This 2nd approach does generate a pattern, albeit not as simple as for the official method.

Solution

(i) Considering the 1st digit, and then the 2nd:

Starting with 44, the next 2 digits must be 44 (total of 1)

Starting with 43, the next 2 digits must be 43 or 34 (total of 2)

Starting with 42, the next 2 digits must be 42 (or 24) or 33 (total of 3)

Starting with 41, the next 2 digits must be 41 (or 14) or 32 (or 23)

(total of 4)

Starting with 40, the next 2 digits must be 40 (or 04) or 31 (or 13) or 22 (total of 5)

[Overall total of 15, so far]

34: next 2 digits must be 43 (or 34) (total of 2)

33: next 2 digits must be 42 (or 24) or 33 (total of 3)

32: next 2 digits must be 41 (or 14) or 32 (or 23) (total of 4)

31: next 2 digits must be 40 (or 04) or 31 (or 13) or 22 (total of 5)

30: next 2 digits must be 30 (or 03) or 21 (or 12) (total of 4)

[Overall total of 15+18=33, so far]

24: 42 (or 24) or 33 (total of 3)

23: 41 (or 14) or 32 (or 23) (total of 4)

22: 40 (or 04) or 31 (or 13) or 22 (total of 5)

21: 30 (or 03) or 21 (or 12) (total of 4)

20: 20 (or 02) or 11 (total of 3)

[Overall total of 33+19=52, so far]

14: 41 (or 14) or 32 (or 23) (total of 4)

13: 40 (or 04) or 31 (or 13) or 22 (total of 5)

12: 30 (or 03) or 21 (or 12) (total of 4)

11: 20 (or 02) or 11 (total of 3)

10: 10 (or 01) (total of 2)

The grand total is then 52+18=70

[An important consideration for this type of question is how quickly you can move through the different cases. Here there is a simple pattern.]

(ii) The generalisation of the above method might arguably need a bit of justification, but applying the same pattern to k in place of 4 gives:

1st digit k: $1 + 2 + \cdots + (k + 1)$

1st digit $k - 1$: $2 + \cdots + (k + 1) + k$

1st digit $k - 2$: $3 + \cdots + (k + 1) + k + (k - 1)$

1st digit $k - 3$: $4 + \cdots + (k + 1) + k + (k - 1) + (k - 2)$

... (k rows)

This can be rearranged as:

$[1 + 2 + \cdots + (k + 1)]$

$[1 + 2 + \cdots + (k + 1)] + (k - 1)$

$[1 + 2 + \cdots + (k + 1)] + 2(k - 2)$

$[1 + 2 + \cdots + (k + 1)] + 3(k - 3)$

... (k rows)

$= k\left[\frac{1}{2}(k + 1)(k + 2)\right] + (k + 2k + 3k + \cdots)[k - 1 \text{ terms}]$

$-(1^2 + 2^2 + 3^2 + \cdots)\ [k - 1 \text{ terms}]$

$= \frac{1}{2}k(k + 1)(k + 2) + k(1 + 2 + \cdots + [k - 1]) - \frac{1}{6}(k - 1)k(2k - 1)$

$= \frac{1}{2}k(k + 1)(k + 2) + k.\frac{1}{2}(k - 1)k - \frac{1}{6}(k - 1)k(2k - 1)$

$= \frac{k}{6}\{3(k + 1)(k + 2) + 3(k - 1)k - (k - 1)(2k - 1)\}$

$$= \frac{k}{6}\{4k^2 + 9k + 5\} = \frac{k}{6}(k+1)(4k+5)$$