STEP 2007, P1, Q13 - Solution (7 pages; 18/3/24)

[This question illustrates 3 possible approaches:

(a) "one step at a time", whereby we imagine the actual sequence of events (considering which disc is taken 1st, then 2nd ...)

(b) $P(B|A) = \frac{P(A \,\&\, B)}{P(A)}$

Note: It is often the case that $P(A \,\&\, B) = P(B)$

(eg if A is the event of rolling an even number on a die, and B is the event of rolling a 2)

(c) As a variation on (b), $\frac{No.\ of\ favourable\ outcomes}{No.\ of\ possible\ outcomes}$

ie $\frac{No.\ of\ \ outcomes\ where\ A\ and\ B\ occur}{No.\ of\ \ outcomes\ \ where\ A\ occurs}$,

provided that the outcomes are equally likely

Notes:

(i) If we are able to count outcomes using numbers of combinations [ie $\binom{n}{r}$], then the outcomes will be equally likely.

(ii) It will normally be easier to deal with numbers of combinations, where order isn't important (rather than considering each possible order).

(i)

Method 1

$$\frac{5}{11}\times\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}=\frac{1}{66}$$

Method 2

$$\frac{No.\ of\ favourable\ outcomes}{No.\ of\ possible\ outcomes}=\frac{\binom{5}{4}}{\binom{11}{4}}=\frac{5}{\left(\frac{11(10)(9)(8)}{4!}\right)}=\frac{1}{66}$$

[Note that, with this method, we are not considering the order in which the discs are taken; we are just looking at the final result; ie that we have selected 4 discs.]

(ii)

Method 1

$P(\text{2nd disc is numbered, given that the 1st disc is number 3})$

$\times\ P(\text{3rd disc is numbered, given that the 1st disc is number 3,}$

$\text{and that the 2nd disc is numbered})\times\ldots$

$$=\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}=\frac{1}{30}$$

Method 2

$$\frac{\text{P(1st disc is number 3 and the other selected discs are numbered)}}{\text{P(1st disc is number 3)}}$$

$$=\frac{\frac{1}{11}\times\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}}{\frac{1}{11}}=\frac{1}{30}$$

Method 3

$$\frac{No.\ of\ favourable\ outcomes}{No.\ of\ possible\ outcomes}=\frac{\binom{4}{3}}{\binom{10}{3}}=\frac{4}{\left(\frac{(10)(9)(8)}{3!}\right)}=\frac{1}{30}$$

[$\binom{10}{3}$ is the number of ways of choosing the remaining 3 items;
$\binom{4}{3}$ is the number of ways of choosing 3 more numbered discs out of the 4 left.]

(iii) **Method 1**

One possibility is $3BNB$ (N is a numbered disc other than 3; B is a blank)

The probability of this occurring, given that the 1st disc is number 3, is $\frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{1}{6}$

As the 2nd numbered disc could be in 3 positions, and the probability is the same in each case,

the required probability $= 3 \times \frac{1}{6} = \frac{1}{2}$

Method 2

$$\frac{P(\text{1st disc is 3 and exactly 1 of the other selected discs is numbered})}{P(\text{1st disc is 3})}$$

$$= \frac{\frac{1}{11} \times [3 \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}]}{\frac{1}{11}} = \frac{1}{2}$$

[The 2nd numbered disc could be in 3 possible positions: 2nd, 3rd or 4th; the probabilities of these are $\frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}$, $\frac{6}{10} \times \frac{4}{9} \times \frac{5}{8}$ & $\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$; ie they are the same.]

Method 3

$$\frac{No.\ of\ favourable\ outcomes}{No.\ of\ possible\ outcomes} = \frac{\binom{4}{1} \times \binom{6}{2}}{\binom{10}{3}} = \frac{4(15)}{\left(\frac{10(9)(8)}{6}\right)} = \frac{1}{2}$$

[Here $\binom{4}{1}$ is the number of ways of choosing the 2nd numbered disc, $\binom{6}{2}$ is the number of ways of choosing the 2 blank discs, and $\binom{10}{3}$ is the number of ways of choosing 3 discs from 10]

(iv) **Method 1**

Whether we know that the disc numbered 3 was taken 1st, or at some other point, makes no difference to the chance of having obtained a 2nd numbered disc. So the probability is still $\frac{1}{2}$.

Method 2

Examples: $N3BB, BNB3$

$N3BB$ has a probability of $\frac{4}{11} \times \frac{1}{10} \times \frac{6}{9} \times \frac{5}{8}$,

and $BNB3$ has a probability of $\frac{6}{11} \times \frac{4}{10} \times \frac{5}{9} \times \frac{1}{8}$

Thus all such cases have the same probability.

The number of cases is 4 [the number of possible places for the 3] $\times$ 3 [the number of possible places for the N] $= 12$

So required probability is $\frac{P(one\ such\ case\ occurs)}{P(3\ occurs)}$

$$= \frac{\frac{4}{11}\times\frac{1}{10}\times\frac{6}{9}\times\frac{5}{8}\times 12}{1-P(3\ doesn't\ occur)} = \frac{(\frac{2}{11})}{1-\frac{10}{11}\times\frac{9}{10}\times\frac{8}{9}\times\frac{7}{8}} = \frac{(\frac{2}{11})}{1-\frac{7}{11}} = \frac{1}{2}$$

Method 3

Required probability is

$$\frac{No.\ of\ ways\ of\ selecting\ a\ 3\ \&\ exactly\ 1\ other\ numbered\ disc}{No.\ of\ ways\ of\ selecting\ a\ 3}\ ,$$

where $No.\ of\ ways\ of\ selecting\ a\ 3$ is

$No.\ of\ ways\ of\ selecting\ 4\ items$

$-\ No.\ of\ ways\ of\ not\ selecting\ a\ 3$

$= \binom{11}{4} - \binom{10}{4}$

and the numerator is $\binom{4}{1}\binom{6}{2}$,

so that required probability is $\frac{\binom{4}{1}\binom{6}{2}}{\binom{11}{4}-\binom{10}{4}} = \frac{4(15)}{\frac{11(10)(9)(8)}{4!}-\frac{10(9)(8)(7)}{4!}}$

$= \frac{4(15)(24)}{(10)(9)(8)[11-7]} = \frac{1}{2}$

(v) [Note here that the situation for (v) & (vi) is different from that of (iii) & (iv). There is a difference between being told that the 1st disc was numbered (when we then have 3 chances to obtain a 2nd disc), and being told that - at the end of the day – it turned out that at least one numbered disc had been taken: in this case the 1st disc may not have been numbered, so the position is not as strong as when we know that the 1st disc was numbered).]

No. of possible outcomes where a numbered disc is taken 1st:

There are 5 ways of choosing a numbered disc for the 1st place, and then $\binom{10}{3}$ ways of filling the remaining places.

No. of favourable outcomes (with exactly 2 numbered discs; one of them being allocated to the 1st place):

There are 5 ways of choosing a numbered disc for the 1st place; then 4 ways of choosing another numbered disc, and then $\binom{6}{2}$ ways of choosing the 2 blank discs.

So $\frac{No.\ of\ favourable\ outcomes}{No.\ of\ possible\ outcomes} = \frac{5\times4\times\binom{6}{2}}{5\times\binom{10}{3}} = \frac{4\times15}{(\frac{10(9)(8)}{3!})} = \frac{1}{2}$

(vi) We can break down the number of possible outcomes, by conditioning on how many numbered discs are taken:

1 taken: $5 \times \binom{6}{3}$ (5 ways of choosing the numbered disc; $\binom{6}{3}$ ways of choosing the 3 blanks)

2 taken: $\binom{5}{2} \times \binom{6}{2}$

3 taken: $\binom{5}{3} \times 6$

4 taken: $\binom{5}{4}$

The total of these is $100 + 150 + 60 + 5 = 315$

[A quicker approach is:

Total number of ways of selecting 4 items [$\binom{11}{4}$],

less number of ways of selecting 4 items, not including a numbered disc [$\binom{6}{4}$]

$= \binom{11}{4} - \binom{6}{4} = \frac{11(10)(9)(8)}{4!} - \binom{6}{2} = 330 - 15 = 315$]

Number of favourable outcomes (where exactly 2 numbered discs are taken) is $\binom{5}{2} \times \binom{6}{2} = 10 \times 15 = 150$ [$\binom{5}{2}$ ways of choosing the 2 numbered discs, and $\binom{6}{2}$ ways of choosing the 2 blank discs]

So $\frac{No.\ of\ favourable\ outcomes}{No.\ of\ possible\ outcomes} = \frac{150}{315} = \frac{30}{63} = \frac{10}{21}$