STEP 2006, P3, Q2 - Solution (3 pages; 17/8/22)

(i) 1st Part

Let $\phi = -\theta$, so that $I = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{cos^2\phi(-d\phi)}{1+sin\phi sin2\alpha} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{cos^2\phi \, d\phi}{1+sin\phi sin2\alpha}$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{cos^2\theta}{1+sin\theta sin2\alpha} \, d\theta$, as required.

2nd Part

Hence $2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{cos^2\theta}{1-sin\theta sin2\alpha} \, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{cos^2\theta}{1+sin\theta sin2\alpha} \, d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{cos^2\theta}{1-sin^2\theta sin^2(2\alpha)}([\,1+sin\theta sin2\alpha] + [1-sin\theta sin2\alpha])d\theta$$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{sec^2\theta - tan^2\theta sin^2(2\alpha)} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{(1+tan^2\theta)-tan^2\theta[1-cos^2(2\alpha)]} \, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{1+tan^2\theta cos^2(2\alpha)} \, d\theta$, as required.

(ii) [It is natural to consider whether a similar method could be employed for J, but this just leads to $J = J$.

Note that $sec^2\theta$ is the derivative of something lurking elsewhere in the integrand – namely $tan\theta$, and that, with the substitution

$u = tan\theta$, $\frac{1}{1+(ucos2\alpha)^2}$ can be integrated. In general, it is usually easiest to spot the derivative of u in the integrand, since it will always be in the numerator.]

Let $u = tan\theta$, so that $du = sec^2\theta \, d\theta$,

and $J = \int_{-\infty}^{\infty} \frac{1}{1+(ucos2\alpha)^2} du = \frac{1}{cos^2(2\alpha)} \int_{-\infty}^{\infty} \frac{1}{sec^2(2\alpha)+u^2} du$

$= sec^2(2\alpha).\frac{1}{\sec(2\alpha)}[tan^{-1}\left(\frac{u}{\sec(2\alpha)}\right)]_{-\infty}^{\infty}$

$= \sec(2\alpha)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \pi\sec(2\alpha)$

$[0 < \alpha < \frac{\pi}{4} \Rightarrow 0 < 2\alpha < \frac{\pi}{2} \Rightarrow \cos(2\alpha) > 0 \Rightarrow \sec(2\alpha) > 0 \Rightarrow$
$\frac{u}{\sec(2\alpha)} > 0$ when $u > 0]$

(iii) $Isin^2(2\alpha) + Jcos^2(2\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{sin^2(2\alpha)}{1+tan^2\theta cos^2(2\alpha)}\, d\theta$

$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{sec^2\theta cos^2(2\alpha)}{1+tan^2\theta cos^2(2\alpha)}\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-cos^2(2\alpha)}{1+tan^2\theta cos^2(2\alpha)}\, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(tan^2\theta+1)cos^2(2\alpha)}{1+tan^2\theta cos^2(2\alpha)}\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1+tan^2\theta cos^2(2\alpha)}\, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{tan^2\theta cos^2(2\alpha)}{1+tan^2\theta cos^2(2\alpha)}\, d\theta$

$= I + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+tan^2\theta cos^2(2\alpha)}{1+tan^2\theta cos^2(2\alpha)}\, d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1+tan^2\theta cos^2(2\alpha)}\, d\theta$

$= I + \left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) - I$ (from the 2nd Part of (i))

$= \pi$

From (ii), $J = \pi\sec(2\alpha)$, so that $Isin^2(2\alpha) + Jcos^2(2\alpha)$

also equals $Isin^2(2\alpha) + \pi\sec(2\alpha)\, cos^2(2\alpha)$

$= Isin^2(2\alpha) + \pi\cos(2\alpha)$

Thus $\pi = Isin^2(2\alpha) + \pi\cos(2\alpha)$,

and $I = \frac{\pi(1-\cos(2\alpha))}{sin^2(2\alpha)} = \frac{\pi(1-[cos^2\alpha - sin^2\alpha])}{sin^2(2\alpha)} = \frac{2\pi sin^2\alpha}{sin^2(2\alpha)}$

$= \frac{2\pi sin^2\alpha}{(2sin\alpha cos\alpha)^2} = \frac{\pi sec^2\alpha}{2}$, as required.

(iv) [You may find that you had overlooked the significance of $0 < \alpha < \frac{\pi}{4}$ in (ii) (it might be necessary to go back and allow for it).]

Once again, $Isin^2(2\alpha) + Jcos^2(2\alpha) = \pi$

$\frac{\pi}{4} < \alpha < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\alpha < \pi \Rightarrow \cos(2\alpha) < 0 \Rightarrow \sec(2\alpha) < 0 \Rightarrow$
$\frac{u}{\sec(2\alpha)} < 0$ when $u > 0$

So now $J = \sec(2\alpha)\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = -\pi\sec(2\alpha)$,

so that $Isin^2(2\alpha) + Jcos^2(2\alpha)$

also equals $Isin^2(2\alpha) - \pi\sec(2\alpha)\,cos^2(2\alpha)$

$= Isin^2(2\alpha) - \pi\cos(2\alpha)$

Hence $\pi = Isin^2(2\alpha) - \pi\cos(2\alpha)$,

and $I = \frac{\pi(1+\cos(2\alpha))}{sin^2(2\alpha)} = \frac{\pi(1+[cos^2\alpha - sin^2\alpha])}{sin^2(2\alpha)} = \frac{\pi(1 - sin^2\alpha + cos^2\alpha)}{sin^2(2\alpha)}$

$= \frac{2\pi cos^2\alpha}{(2sin\alpha cos\alpha)^2} = \frac{\pi cosec^2\alpha}{2}$