STEP 2006, Paper 3, Q12 – Solution (2 pages; 20/5/18)

Let X be the number of tourists (out of 1024) taking the bus.

Then $X \sim B(1024, \frac{1}{2}) \approx N(512, 256)$

E(Annual profit | n buses) $= 50P(X \geq 32n)(32n)$

$+50\int_{-\infty}^{32n} xf(x)dx = A(n) + B(n)$, say

The largest licence fee to be considered is the extra profit to be expected from increasing the number of buses from 15 to 16; ie

$A(16) + B(16) - [A(15) + B(15)]$

$P(X \geq 32n) = 1 - P(Z < \frac{32n-512}{\sqrt{256}})$

$P(X \geq 32(16)) = 1 - P(Z < 0) = 0.5$

$P(X \geq 32(15)) = 1 - P(Z < -2) = P(Z < 2) = \Phi(2)$

So $A(16) = 50(32)(16)(0.5) = 12800$

and $A(15) = 50(32)(15)\Phi(2) = 24000\Phi(2)$

$B(16) - B(15) = 50\int_{480}^{512} xf(x)dx$

Let $z = \frac{x-512}{\sqrt{256}} = \frac{x-512}{16}$, so that $dz = \frac{1}{16}dx$,

and since $f(x) = \frac{1}{16\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-512}{16})^2}$,

$B(16) - B(15) = 50\int_{-2}^{0}(16z + 512)\frac{1}{16\sqrt{2\pi}}e^{-\frac{1}{2}z^2}(16)dz$,

$= 800\int_{-2}^{0} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz + 25600\int_{-2}^{0}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$

$= 800\frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{1}{2}z^2}\right]_{-2}^{0} + 25600(\Phi(2) - 0.5)$

$= 800\frac{1}{\sqrt{2\pi}}(-1 + e^{-2}) + 25600\Phi(2) - 12800$

Thus $A(16) + B(16) - [A(15) + B(15)]$

$= A(16) - A(15) + [B(16) - B(15)]$

$= [12800 - 24000\Phi(2)] + 800\frac{1}{\sqrt{2\pi}}(e^{-2} - 1)$

$+25600\Phi(2) - 12800$

$= 1600\Phi(2) - 800\frac{1}{\sqrt{2\pi}}(1 - e^{-2})$, as required