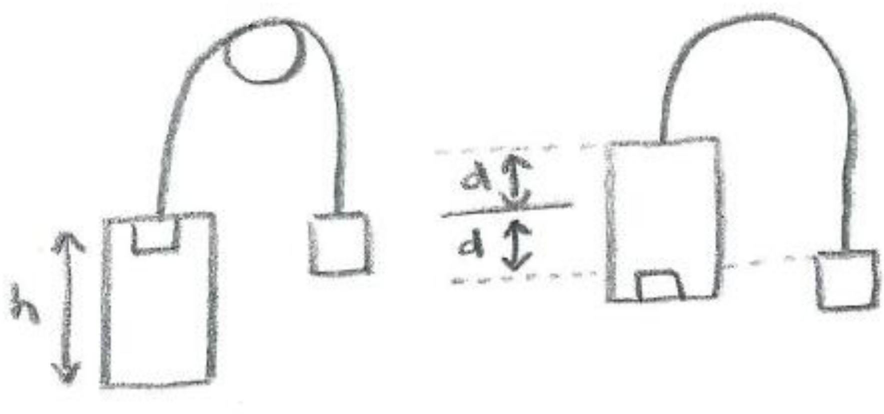
STEP 2006, Paper 3, Q11 – Solution (6 pages; 20/5/18)



[As an alternative to the relative motion approach adopted in the H&As:]

This problem concerns 3 objects: the lift (excluding the detached tile) (L), the tile (T) and the counterweight (C).

Suppose that, in the time taken (t, say) for the tile to reach the floor of the lift, the lift and the counterweight have both moved by a distance d, so that the tile has moved a distance $h - d$.

The information available, or to be found, can be gathered together in the following table:

	T	L / C
distance moved	$h - d$	d
initial speed	0	0
final speed	v	
accel.	g	a
time	t	t

Applying "$s = ut + \frac{1}{2}at^2$",

$T: h - d = \frac{1}{2}gt^2$; $L: d = \frac{1}{2}at^2$ (1)

Applying $v^2 = u^2 + 2as$ for T: $v^2 = 2g(h - d)$ (2)

Then, from (1), $t^2 = \frac{2(h-d)}{g}$ & $t^2 = \frac{2d}{a}$,

so that $\frac{2(h-d)}{g} = \frac{2d}{a}$; $ah - ad = gd$, and $d = \frac{ah}{a+g}$

Then from (2), $v^2 = 2g\left(h - \frac{ah}{a+g}\right) = \frac{2gh(a+g-a)}{a+g} = \frac{2g^2h}{a+g}$ (3)

We then find a by the usual method:

If F is the tension in the cable, N2L applied separately to the lift and the counterweight gives:

$L: F - (M - m)g = (M - m)a$

$C: Mg - F = Ma$

Adding these equations then gives $mg = (2M - m)a$,

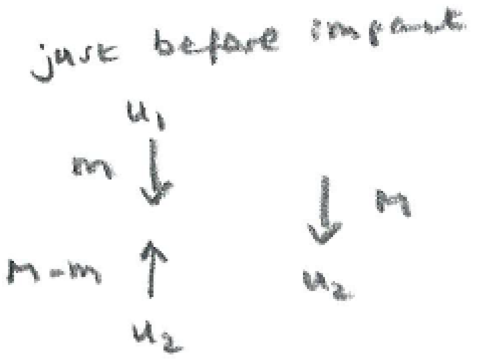
so that $a = \frac{mg}{2M-m}$ and, from (3), $v^2 = \frac{2g^2h}{a+g} = \frac{2g^2h}{\left(\frac{mg}{2M-m}+g\right)}$

$= \frac{(2M-m)(2g^2h)}{mg+2Mg-mg} = \frac{(2M-m)gh}{M}$

and hence $v = \sqrt{\frac{(2M-m)gh}{M}}$, as required.

For the 2nd part, it isn't immediately clear whether we can make any assumption about the direction of the lift and the counterweight after the tile has bounced off the floor of the lift. The H&As assume in the diagram that the directions of the lift and counterweight are reversed after the impact, but for our diagram we can maintain the original directions (which will then produce negative figures if the directions are in reality reversed).

Also, we need to decide what to make of the information about the impulsive forces. Where this sort of information is provided, it is often the case that no special action is required: the information is just there to reassure us that nothing unusual applies (just as the pulley is described as being fixed and frictionless). Of course, it could be the case that the information is adding a constraint to the system which wouldn't normally apply (in the same way that, for a car on a banked track, we might be told that there was no friction). In this question, as the examiner's report points out, the information could have been assumed anyway, but is mentioned as a hint.



Applying conservation of momentum: [This is suggested by the mention of impulses. It also has the advantage of avoiding squared velocity terms. The fact that we are trying to find a loss of energy rules out the use of conservation of energy.]

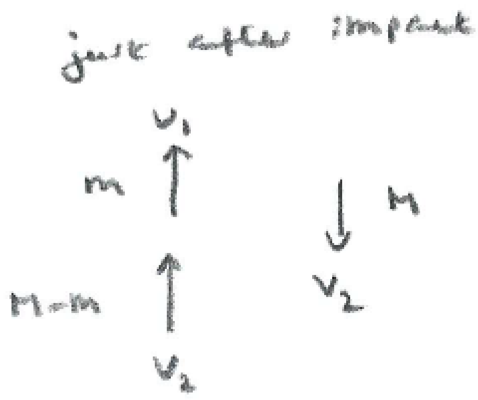
Just before impact of tile with floor of lift

$Mu_2 = mu_1 - (M - m)u_2$

$\Rightarrow u_2(2M - m) = mu_1$

$\Rightarrow u_2 = \alpha u_1$, where $\alpha = \frac{m}{2M-m}$ (4)

(Note that u_1 was determined in the 1st part of the question, as v.)



Just after impact

When the tile hits the floor, there will be an impulsive force downwards on the cable attaching to the lift, and hence - by N3L - an equal impulsive force upwards on the lift (and tile). The downwards impulsive force on the cable (attaching to the lift) means that an equal impulsive force (upwards) is being applied to the counterweight (as indicated in the question). The implication of this is that the counterweight loses momentum in the downwards direction (though we don't know whether this is sufficient to reverse its direction of motion) and that this loss of momentum is balanced by an increase in momentum (upwards) of the lift and tile. Thus the total momentum in the original direction of motion of the lift and counterweight is unchanged (which could have been stated without any reference to impulsive forces!) Throughout the motion, the lift and counterweight must of course be moving with the same speed and direction.

Loss of momentum (downwards) of counterweight = gain in momentum (upwards) of the lift and tile:

$M(u_2 - v_2) = (M - m)(v_2 - u_2) + m(v_1 - (-u_1))$

$\Rightarrow u_2(2M - m) = v_2(2M - m) + mv_1 + mu_1$

$\Rightarrow \alpha u_1 \left(\frac{m}{\alpha}\right) = v_2 \left(\frac{m}{\alpha}\right) + mv_1 + mu_1$, from (4)

$\Rightarrow 0 = v_2 \left(\frac{m}{\alpha}\right) + mv_1 \Rightarrow v_2 = -\alpha v_1$

(This indicates that the lift and counterweight have in fact reversed the direction of their motion, assuming that the tile is moving upwards after hitting the floor - which must be the case: if it were moving downwards (so that v_1 is negative), then the lift would also have to be moving downwards, so that v_2 would be negative; but $v_2 = -\alpha v_1$ implies that v_2 is positive; ie a contradiction.)

Then from Newton's law of restitution:

$v_1 - v_2 = e(u_2 - (-u_1))$

so that $v_1 - (-\alpha v_1) = e(\alpha u_1 + u_1)$

$\Rightarrow v_1(1 + \alpha) = eu_1(\alpha + 1) \Rightarrow v_1 = eu_1$

Thus all the speeds have been found in terms of u_1.

The loss of energy caused by the impact is the instantaneous loss of kinetic energy (since the potential energy doesn't change instantaneously).

Hence loss of energy

$= \frac{1}{2}(M - m)({u_2}^2 - {v_2}^2) + \frac{1}{2}m({u_1}^2 - {v_1}^2) + \frac{1}{2}M({u_2}^2 - {v_2}^2)$

$= \frac{1}{2}(2M - m)((\alpha u_1)^2 - (-\alpha e u_1)^2) + \frac{1}{2}m({u_1}^2 - (eu_1)^2)$

$= \frac{1}{2}{u_1}^2\{\frac{m}{\alpha}(\alpha^2 - \alpha^2 e^2) + m(1 - e^2)\}$

$= \frac{1}{2}m{u_1}^2(1 - e^2)(\alpha + 1)$

Now $\alpha + 1 = \frac{m}{2M-m} + \frac{2M-m}{2M-m} = \frac{2M}{2M-m}$

and ${u_1}^2 = \frac{(2M-m)gh}{M}$, from the 1st part

So loss of energy $= \frac{m(2M-m)gh(1-e^2)(2M)}{2M(2M-m)} = mgh(1 - e^2)$,

as required.

In the case of an object being dropped from a height of h onto the ground, its kinetic energy before impact is $\frac{1}{2}mv^2 = mgh$ (by conservation of energy), and its kinetic energy after rebounding is $\frac{1}{2}m(ev)^2 = e^2(mgh)$, so that the loss of energy is $mgh(1 - e^2)$;

ie the motion of the lift makes no difference to the energy lost.

[It would be quite an achievement to complete this question in the 45 minutes allotted!]