STEP 2006, P2, Q6 – Solution (3 pages; 17/5/18)

This is a strange question in a number of ways, and it is possible that it has been changed (without modifying the Hints & Answers), since the H&As state an answer of "$x = 7$" to (i), which doesn't fit the question.

Let $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ & $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $u.v = |u||v|cos\theta$, where θ is the angle between u & v.

As $|cos\theta| \leq 1$, $(u.v)^2 \leq |u|^2|v|^2$,

so that $(ax + by + cz)^2 \leq (a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2)$, as required.

The instruction to 'deduce' necessary and sufficient conditions for the equality to hold seems a bit misleading: the H&A just refers to the fact that $cos\theta = 1 \Leftrightarrow u$ & v are parallel; ie $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, but this doesn't seem to be a deduction from the inequality as such.

It is also possible to expand both sides of the equation, as follows (though, as pointed out in the Examiner's Report, this definitely isn't deducing anything from the inequality, and so would generally not get any credit):

$(ax + by + cz)^2 = (a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2)$

$\Leftrightarrow a^2x^2 + b^2y^2 + c^2z^2 + 2(axby + axcz + bycz)$

$= a^2x^2 + b^2y^2 + c^2z^2 + (a^2y^2 + a^2z^2 + b^2x^2 + b^2z^2 + c^2x^2 + c^2y^2)$

$\Leftrightarrow (ay - bx)^2 + (az - cx)^2 + (bz - cy)^2 = 0$

$\Leftrightarrow ay - bx = 0; az - cx = 0; bz - cy = 0$

$\Leftrightarrow \frac{a}{x} = \frac{b}{y}$; $\frac{a}{x} = \frac{c}{z}$; $\frac{b}{y} = \frac{c}{z}$; ie $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$

For (i), the required result follows from just setting $a = 1, b = 2$ & $c = 2$ in the original inequality.

For (ii), it seems that the earlier equality is more likely to be relevant than the inequality in (i) [which is worrying, as (ii) is then not being used].

If (which isn't at all clear), the equality **is** to be used, there isn't an obvious correspondence between the equality and the equations to be satisfied.

We can see that $729 = 9 \times 81 = 3^6$ & $243 = 81 \times 3 = 3^5$, which could be useful.

If there is to be any connection with the equality, it seems likely that

$p^2 + 4q^2 + 9r^2$ needs to be written as $a^2 + b^2 + c^2$ [or $x^2 + y^2 + z^2$],

ie setting $a = p, b = 2q$ & $c = 3r$

We can then investigate the other equation, which becomes

$8a + 4b + c = 243$

The LHS of this equation is presumably to be matched with $ax + by + cz$,

giving $x = 8, y = 4$ & $z = 1$

These substitutions are of course only being made in the hope that the earlier equality may be of use.

Making these substitutions, we get:

$(8p + 8q + 3r)^2 = (p^2 + 4q^2 + 9r^2)(64 + 16 + 1)$ (A)

when $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$; ie when $\lambda = \frac{p}{8} = \frac{2q}{4} = \frac{3r}{1}$

We want $8p + 8q + 3r = 243$, so that $64\lambda + 16\lambda + \lambda = 243$, and hence

$\lambda = \frac{243}{81} = 3$ (reassuringly)

So with $p = 8\lambda = 24$, $q = \frac{4\lambda}{2} = 6$ & $r = \frac{\lambda}{3} = 1$, (A) says that

$p^2 + 4q^2 + 9r^2 = \frac{243^2}{81} = \frac{(3^5)^2}{3^4} = 3^6 = 729$, as required.