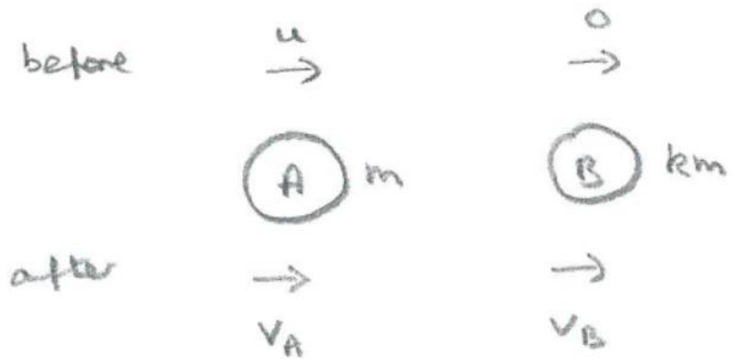
STEP 2006, Paper 2, Q10 - Solution (4 pages; 13/4/21)

(i)



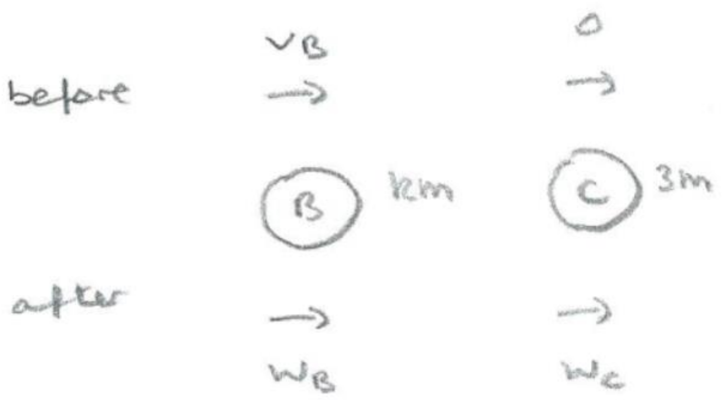
Referring to the diagram, by conservation of momentum:

$mu = mv_A + kmv_B$, so that $u = v_A + kv_B$ (1)

And, by Newton's law of restitution, $v_B - v_A = \frac{1}{2}u$ (2)

Adding (1) & (2) gives $(k+1)v_B = \frac{3u}{2}$, so that $v_B = \frac{3u}{2(k+1)}$

Then, from (2), $v_A = \frac{3u}{2(k+1)} - \frac{1}{2}u = \frac{u(3-k-1)}{2(k+1)} = \frac{u(2-k)}{2(k+1)}$



For the collision between B and C,

CoM: $kmv_B = kmw_B + 3mw_C$,

so that $kv_B = kw_B + 3w_C$ (3)

NLR: $w_C - w_B = \frac{1}{4}v_B$ (4)

Substituting for w_C from (4) into (3),

$kv_B = kw_B + 3(w_B + \frac{1}{4}v_B)$

$\Rightarrow 4kv_B = 4kw_B + 12w_B + 3v_B$

$\Rightarrow v_B(4k-3) = w_B(4k+12)$

$\Rightarrow w_B = \frac{v_B(4k-3)}{4(k+3)} = \frac{3u(4k-3)}{8(k+1)(k+3)}$

We need to investigate under what circumstances $v_A > w_B > 0$ (when A and B are both moving to the right), or

$-w_B > -v_A > 0$ (when they are both moving to the left); ie

$0 > v_A > w_B$

So the required condition is just $v_A > w_B$

ie $\frac{u(2-k)}{2(k+1)} > \frac{3u(4k-3)}{8(k+1)(k+3)}$

or $4(2-k)(k+3) > 3(4k-3)$

or $4k^2 + 16k - 33 < 0$

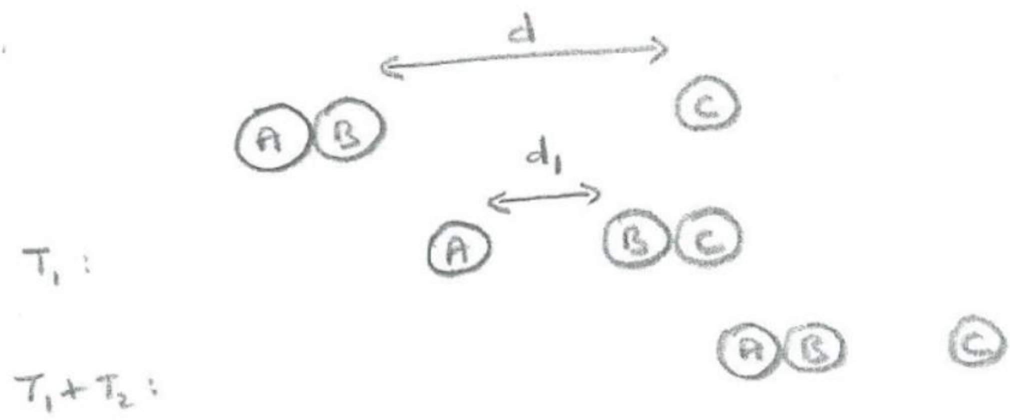
or $(2k-3)(2k+11) < 0$

(noting that $4 \times 33 = (2 \times 3) \times (2 \times 11)$, and that $22 - 6 = 16$; and then that the $22k$ can only be obtained from $2k(11)$)

Hence, considering the quadratic curve, $-\frac{11}{2} < k < \frac{3}{2}$,

and thus, as $k > 0$, $0 < k < \frac{3}{2}$

(ii)



Let T_1 be the period between the 1st collision of A & B and the collision of B & C. Then $T_1 = \frac{d}{v_B}$

From (i), $v_B = \frac{3u}{2(k+1)}$, so that with $k = 1, T_1 = \frac{4d}{3u}$

With $k = 1, v_A = \frac{u(2-k)}{2(k+1)} = \frac{u}{4}$

During T_1, A and B have relative speeds $v_B - v_A = \frac{1}{2}u$, from (i).

So during this period A and B have moved apart by a distance

$d_1 = \frac{1}{2}uT_1 = \frac{1}{2}u \cdot \frac{4d}{3u} = \frac{2d}{3}$

With $k = 1, w_B = \frac{3u(4k-3)}{8(k+1)(k+3)} = \frac{3u}{64}$

Let T_2 be the period between the collision of B & C and the 2nd collision of A & B.

During this time, A and B have relative speeds $w_B - v_A$

$= \frac{3u}{64} - \frac{u}{4} = \frac{-13u}{64}$

and so $T_2 = \frac{d_1}{(\frac{13u}{64})} = \frac{(\frac{2d}{3})}{(\frac{13u}{64})} = \frac{128d}{39u}$

Hence the required time, $T_1 + T_2 = \frac{4d}{3u} + \frac{128d}{39u}$

$= \frac{d(52+128)}{39u} = \frac{180d}{39u} = \frac{60d}{13u}$, as required.