STEP 2006, Paper 1, Q6 – Solution (2 pages; 13/5/18)

(i) $(3a+4b)^2 - 2(2a+3b)^2$

$= a^2(9-8) + b^2(16-18) + ab(24-24)$

$= a^2 - 2b^2 = 1$, as required

(ii) We want $M(5a+6b)^2 - N(4a+5b)^2 = 1 \;\; (A)$, given that

$Ma^2 - Nb^2 = 1$

Now (A) $\Leftrightarrow$

$a^2(25M - 16N) + b^2(36M - 25N) + 2ab(30M - 20N) = 1$

$\Leftrightarrow 25(Ma^2 - Nb^2) - 16Na^2 + 36Mb^2 + 2ab(30M - 20N) = 1$

$\Leftrightarrow 16Na^2 - 36Mb^2 - 2ab(30M - 20N) = 24 \quad (B)$,

as $Ma^2 - Nb^2 = 1$

Suppose for the moment that $30M - 20N = 0$, so that $M = \frac{2}{3}N$

Then (B) $\Leftrightarrow 24Ma^2 - 24Nb^2 = 24$,   which is true,

as $Ma^2 - Nb^2 = 1$

Thus, if $M = \frac{2}{3}N$, (A) is true. So $M = 2, N = 3$ will satisfy the requirement.

[The following outline of an argument is probably not a good use of time in the exam, as it is highly likely that this issue will be glossed over from a marking point of view (it isn't considered in the H&As), and in any case outlines are not usually sufficient. The full argument would take much too long.

In order to investigate the existence of other solutions which could be described as 'smaller', suppose that instead $M = \lambda N$ (where $\lambda \neq 2/3$).

Then (B) becomes

$16\left(\frac{M}{\lambda}\right)a^2 - 36(\lambda N)b^2 - 2ab(30(\lambda N) - 20N) = 24$

or $\frac{16}{\lambda}(Ma^2 - Nb^2) + Nb^2(\frac{16}{\lambda} - 36\lambda) - 20abN(3\lambda - 2) = 24$

and hence $\frac{16}{\lambda} + Nb^2(\frac{16}{\lambda} - 36\lambda) - 20abN(3\lambda - 2) = 24$

We could then substitute for a from $Ma^2 - Nb^2 = 1$

(with $M = \lambda N$), to show that, for a given λ, N depended on b, and of course there are an infinite number of values of b satisfying

$Ma^2 - Nb^2 = 1$. In fact, we would only need to show this for

$\lambda = 2$ & $\lambda = \frac{1}{2}$, as M & N are to be the smallest possible positive integers.]

(iii) We want $(Pa + Qb)^2 - 3(Ra + Sb)^2 = 1$ (C), given that

$a^2 - 3b^2 = 1$

$(C) \Leftrightarrow a^2(P^2 - 3R^2) + b^2(Q^2 - 3S^2) + 2ab(PQ - 3RS) = 1$ (D)

Let $PQ - 3RS = 0$, and suppose that $P = S$, so that $Q = 3R$

Then (D) $\Leftrightarrow$ $P^2(a^2 - 3b^2) - 3a^2R^2 + 9b^2R^2 = 1$

and hence $P^2 - 3(a^2 - 3b^2)R^2 = 1$

and $P^2 - 3R^2 = 1$

For example, $P = 2, R = 1$, with $S = 2$ & $Q = 3$