STEP 2006, Paper 1, Q5 – Solution (2 pages; 13/5/18)

(i) $u^2 = 2x + 1 \Rightarrow 2u\,du = 2\,dx$ and $x - 4 = \frac{u^2-1}{2} - 4 = \frac{u^2-9}{2}$

and $\int \frac{3}{(x-4)\sqrt{2x+1}} dx = 3\int \frac{u}{\left(\frac{u^2-9}{2}\right)u} du = 6\int \frac{1}{(u-3)(u+3)} dx$

$\int \frac{1}{u-3} - \frac{1}{u+3} du = \ln|u-3| - \ln|u+3| + K = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K$,

as $\sqrt{2x+1} - 3 > 0$ when $x > 4$

(ii) Let $u^2 = e^x + 1$ (where $u > 0$), so that $2u\,du = e^x\,dx$

and $dx = \frac{2u}{u^2-1}\,du$

Then $\int_{ln3}^{ln8} \frac{2}{e^x\sqrt{e^x+1}}\,dx = 2\int_2^3 \frac{1}{(u^2-1)u}\left(\frac{2u}{u^2-1}\right) du$

$= 4\int_2^3 \frac{1}{(u^2-1)^2} du = 4\int_2^3 \frac{1}{(u-1)^2(u+1)^2} du$

$= 4\int_2^3 \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2} du$

where $1 = A(u-1)(u+1)^2 + B(u+1)^2$

$+C(u+1)(u-1)^2 + D(u-1)^2$ (for all u)

Setting $u = 1$: $1 = 4B$ and hence $B = \frac{1}{4}$

Setting $u = -1$: $1 = 4D$ and hence $D = \frac{1}{4}$

Setting $u = 0$: $1 = -A + B + C + D$, so that $C - A = \frac{1}{2}$

Then equating coeffs of u^3: $0 = A + C$

Hence $C = \frac{1}{4}$ & $A = -\frac{1}{4}$

So $4\int_2^3 \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2} du$

$= \int_2^3 \frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\, du$

$= \left[- \ln(u-1) - \frac{1}{u-1} + \ln(u+1) - \frac{1}{u+1}\right]_2^3$

(since $u - 1$ & $u + 1$ are both positive)

$= \left(-ln2 - \frac{1}{2} + ln4 - \frac{1}{4}\right) - (0 - 1 + ln3 - \frac{1}{3})$

$= ln\left(\frac{4}{2(3)}\right) + \frac{7}{12} = \frac{7}{12} + ln\left(\frac{2}{3}\right)$, as required