STEP 2006, Paper 1, Q1 – Solution (4 pages; 12/5/18)

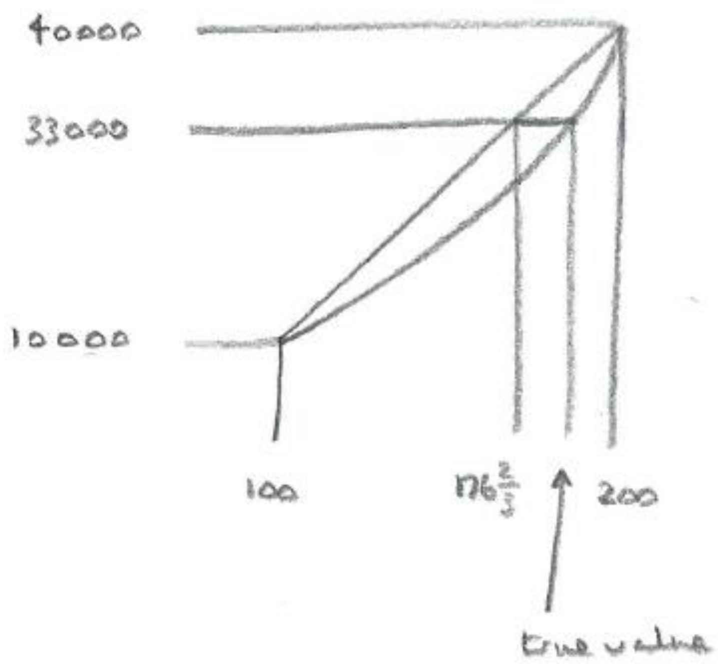
[Squaring 3 digit numbers is obviously a bit time-consuming without a calculator. There are a couple of devices that can be used here though: namely linear interpolation and difference of two squares.]

First of all, $100^2 = 10000 \ \& \ 200^2 = 40000$

Linear interpolation gives an approximate value for x such that

$x^2 = 33000$: $\quad x \approx 100 \times \frac{7}{30} + 200 \times \frac{23}{30} = \frac{70+460}{3} = \frac{530}{3} = 176\frac{2}{3}$

Because of the convex shape of $y = x^2$, $\frac{530}{3}$ will be an underestimate for x (see diagram below).



So we could perhaps try 180 next:

$180^2 = 100(100 + 64 + 160) = 32400$

Note now that $181^2 - 180^2 = (181 + 180)(181 - 180) = 361$

and that $182^2 - 181^2 = (182 + 181)(1) = 363$

So $182^2 = 32400 + 361 + 363 = 32400 + 724 = 33124$

(and $183^2 = 33124 + 365$)

Thus $182^2 < 33127 < 183^2$; ie $n = 182$

Check: $182^2 = (100 + 80 + 2)^2$

$= 10000 + 6400 + 4 + 2(8000 + 200 + 160)$

$= 16404 + 2(8360) = 16404 + 16720 = 33124$

From the earlier working,

$183^2 - 33127 = (33124 + 365) - 33127 = 362$

$184^2 - 33127 = (33124 + 365 + 367) - 33127$

$= 362 + 367 = 729 = 27^2$

ie $m = 2$

So $184^2 - 33127 = 27^2$,

and hence $33127 = 184^2 - 27^2 = (184 + 27)(184 - 27)$

$= 211(157)$

Suppose that $(182 + m)^2 - 33127 = r^2$ (where r is an integer),

so that $33127 = (182 + m)^2 - r^2 = (182 + m + r)(182 + m - r)$

We need to see if 211 and 157 have any factors:

211 is found to be prime (we only have to consider factors up to 15, as $15^2 > 211$, and so a factor greater than 15 requires another one less than 15), as is 157 (considering factors up to 13).

$33127 = (182 + m + r)(182 + m - r)$ then implies that one of the following is true:

(i) $182 + m + r = 211$ & $182 + m - r = 157$,

giving $m = 2$ & $r = 27$, as above

(ii) $182 + m + r = 157$ & $182 + m - r = 211$,

which is the same as (i), except that r is replaced by $-r$;

so once again $m = 2$ (and $r = -27$)

(iii) $182 + m + r = 33127$ & $182 + m - r = 1$,

in which case $m + r = 32945$ & $m - r = -181$,

so that $2m = 32764$ & $m = 16382$

(iv) $182 + m + r = 1$ & $182 + m - r = 33127$,

which is the same as (iii), except that r is replaced by $-r$;

so once again $m = 16382$

Thus there are only two values for m and the other value is 16382.

[In the official solutions, the factorisation

$33127 = (16564 - 16563)(16564 + 16563)$ is (presumably)

arrived at by noting that $(182 + m) - r = 1$, with

$(182 + m) + r = 33127$ (ie (iii) above), so that $(182 + m)$ & r

differ by 1 and add to give 33127; it is then just a matter of halving 33127 to give $33127 = 2(16563) + 1$, so that

$33127 = 16564 + 16563$; then $182 + m = 16564$

and hence $m = 16382$]