STEP 2006, Paper 1, Q14 – Solution (2 pages; 16/5/18)

(i) If X is the number of draws needed in order to obtain the 1st red sweet, then $X \sim Geo(p)$, where p, the probability of success (ie drawing a red), is $\frac{1}{n+1}$

So Prob$(X = r) = \left(\frac{n}{n+1}\right)^{r-1}\left(\frac{1}{n+1}\right) = \frac{n^{r-1}}{(n+1)^r}$

Stationary point(s) occur when

$\frac{d}{dn}\frac{n^{r-1}}{(n+1)^r} = 0$;

so that $\frac{(n+1)^r(r-1)n^{r-2} - n^{r-1}(r)(n+1)^{r-1}}{(n+1)^{2r}} = 0$ (A)

$\Rightarrow (n+1)^{r-1}n^{r-2}\{(n+1)(r-1) - nr\} = 0$ (B)

$\Rightarrow -n + r - 1 = 0 \Rightarrow n = r - 1$

We need to rule out the possibilities of a minimum or a point of inflexion [though the official sol'n doesn't mention this].

The fact that $\frac{n^{r-1}}{(n+1)^r} \to 0\ as\ n \to \infty$ rules out a (local) minimum or a point of inflexion of the same shape as $y = x^3$, as either of these would imply the existence of a maximum for a larger value of n, which contradicts the fact that there is only one stationary point.

In order to rule out a point of inflexion of the same shape as

$y = -x^3$, we can examine the sign of $\frac{d}{dn}\frac{n^{r-1}}{(n+1)^r}$ at $n = r - 2$:

From (A) & (B), the required sign will be that of

$\{(n+1)(r-1) - nr\}$ when $n = r - 2$;

ie of $(r-1)(r-1) - (r-2)r = -2r + 1 + 2r = 1$

So $\frac{d}{dn}\frac{n^{r-1}}{(n+1)^r} > 0$ to the left of $n = r - 1$, which rules out a point of inflexion of the same shape as $y = -x^3$ (unless there was another maximum to the left, which gives a contradiction, as before).

Hence the probability is maximised when $n = r - 1$

[As a check for reasonableness, consider two extreme situations:

A: $r = 5$ and B: $r = 100$

For A, a large n such as 100 would give a much smaller probability than $n = 4$

For B, the probability of taking 100 goes to obtain the 1st red is much smaller for say $n = 5$ than for $n = 99$]

(ii) Prob$(X = r) = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\ldots\left(\frac{n-[r-2]}{n-[r-2]+1}\right)\left(\frac{1}{n-[r-2]}\right)$

$= \frac{1}{n+1}$

This is maximised when n is minimised, and it's tempting to say that this occurs when $n = 0$ (there's also the issue of whether there can be no blue sweets in the bag (there probably could)).

However there is another constraint: $r \leq n + 1$;ie $n \geq r - 1$,

so that n is minimised when $n = r - 1$

Approach 2: as in the official solutions