STEP 2006, Paper 1, Q13 – Solution (2 pages; 16/5/18)

[Note that the Poisson distribution is in the STEP 1&2 syllabus.]

(i) P(0 diamonds | N rolled) $= e^{-\frac{N}{10}}$,

as number of diamonds $\sim Po(\frac{N}{10})$

So P(0 diamonds) $= \sum_{N=1}^{6} P(N\ rolled)$P(0 diamonds | N rolled)

$= \sum_{N=1}^{6} \frac{1}{6} e^{-\frac{N}{10}}$

$= \frac{1}{6} e^{-\frac{1}{10}} \frac{(1-e^{-\frac{6}{10}})}{1-e^{-\frac{1}{10}}}$ (the sum of a Geometric series), as required

E(number of diamonds) =

$\sum_{N=1}^{6} P(N\ rolled)E(number\ of\ diamonds|N\ rolled)$

$= \sum_{N=1}^{6} \frac{1}{6} (\frac{N}{10}) = \frac{1}{60}\left(\frac{1}{2}\right)(6)(7) = \frac{7}{20} = 0.35$, as required

(ii) P(1st 6 on Tth throw) $= \left(\frac{5}{6}\right)^{T-1}\left(\frac{1}{6}\right)$

So P(0 diamonds) $= \sum_{T=1}^{\infty} \left(\frac{5}{6}\right)^{T-1}\left(\frac{1}{6}\right) e^{-\frac{T}{10}}$

$= \frac{1}{6}\left(\frac{5}{6}\right)^{-1} \sum_{T=1}^{\infty} \left(\frac{5}{6}\right)^{T} e^{-\frac{T}{10}}$

$= \frac{1}{5} \sum_{T=1}^{\infty} \left(\frac{5}{6} e^{-\frac{1}{10}}\right)^{T}$

$= \frac{1}{5}\left(\frac{5}{6} e^{-\frac{1}{10}}\right) \frac{1}{1-\frac{5}{6}e^{-\frac{1}{10}}}$

$= \frac{e^{-\frac{1}{10}}}{6-5e^{-\frac{1}{10}}}$, as required

E(number of diamonds) $= \sum_{T=1}^{\infty}\left(\frac{5}{6}\right)^{T-1}\left(\frac{1}{6}\right)\left(\frac{T}{10}\right)$

$= \frac{1}{60}\sum_{T=1}^{\infty}\left(\frac{5}{6}\right)^{T-1} T$

[$(1-x)^{-2}$ will be useful because it is the derivative of

$(1-x)^{-1} = 1 + x + x^2 + \cdots$, and will therefore have terms of the form Tx^{T-1}]

Consider $(1-x)^{-2} =$

$1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \cdots$

$= 1 + 2x + 3x^2 + 4x^3 + \cdots$

Thus E(number of diamonds) $= \frac{1}{60}(1-\frac{5}{6})^{-2}$

$= \frac{36}{60} = \frac{6}{10} = 0.6$