STEP 2005, Paper 2, Q4 - Solutions (4 pages; 11/5/18)

Let $\frac{1}{a+b} = tan\theta$, $\frac{1}{a+c} = tan\phi$, $\frac{1}{a} = tan\alpha$,

where θ, ϕ & $\alpha \in (0, \frac{\pi}{2})$, since the range of $arctan$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$ and a, b & c are >0, so that $tan\theta, tan\phi$ & $tan\alpha$ are >0, and hence θ, ϕ & α are >0

rtp [result to prove]: $\theta + \phi = \alpha$

$\tan(\theta + \phi) = tan\alpha \Rightarrow \theta + \phi = \alpha + n\pi$ (for integer n) (1)

But $\theta + \phi < \frac{\pi}{2} + \frac{\pi}{2} = \pi$, so that $n \leq 0$, from (1), since $\alpha > 0$

[$n = 1$, for example, leads to a contradiction]

whilst $\theta + \phi > 0$, so that $n \geq 0$, from (1), since $\alpha < \pi$

[$n = -1$, for example, leads to a contradiction]

Thus $n = 0$, and we just need to show that $\tan(\theta + \phi) = tan\alpha$

Now $\tan(\theta + \phi) = \frac{tan\theta + tan\phi}{1 - tan\theta tan\phi} = \frac{\frac{1}{a+b} + \frac{1}{a+c}}{1 - \left(\frac{1}{a+b}\right)\left(\frac{1}{a+c}\right)}$

$= \frac{(a+c)+(a+b)}{(a+b)(a+c)-1} = \frac{2a+b+c}{a^2+ac+ab+bc-1}$ (A)

[we want this to be $\frac{1}{a}$, so a needs to be a factor of the denominator]

As $bc - 1 = a^2$ (given),

$(A) = \frac{2a+b+c}{a^2+ac+ab+a^2} = \frac{2a+b+c}{a(2a+b+c)} = \frac{1}{a}$

[as this is effectively a 'show that' result, we have to be wary of jumping to the answer]

[For the next part, we can expect to do something related to the 1st part. It might be to extend the method to $\tan(\theta+\phi+\beta+\gamma)$, but another use of the 1st part is just to apply its result in the 2nd part (which is simpler - and usually with STEP q'ns it does turn out to be something simple).]

Let $a = p + q$, $b = s$ & $c = t$, and apply the result that has just been proved, since $bc = st = (p+q)^2 + 1 = a^2 + 1$, as required.

Thus $\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) = \arctan(\frac{1}{p+q})$

and, in exactly the same way,

$\arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right) = \arctan(\frac{1}{p+r})$

and $\arctan\left(\frac{1}{p+q}\right) + \arctan\left(\frac{1}{p+r}\right) = \arctan(\frac{1}{p})$,

so that $\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) + \arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right)$

$= \arctan\left(\frac{1}{p+q}\right) + \arctan(\frac{1}{p+r}) = \arctan(\frac{1}{p})$

Then let $p = 7$

and suppose that

$q + s = 13 - 7 = 6$ (i)

$q + t = 21 - 7 = 14$ (ii)

$r + u = 82 - 7 = 75$ (iii)

$r + v = 187 - 7 = 180$ (iv)

with $st = (7+q)^2 + 1$ (v)

$uv = (7+r)^2 + 1$ (vi)

& $qr = 50$ (vii)

[Note that we have 7 equations in only 6 unknowns, so we will need to ensure that the solutions are consistent with ALL the equations.]

[Also, (i), (ii) & (v) involve just 3 of the unknowns, and similarly for (iii), (iv) & (vi)]

$q + s = 6$ (i)

$q + t = 14$ (ii)

$st = (7 + q)^2 + 1$ (v)

Eliminating s & t from (i) & (ii) gives

$(6 - q)(14 - q) = (7 + q)^2 + 1$

$\Rightarrow 84 - 20q = 50 + 14q$

$\Rightarrow 34q = 34 \Rightarrow q = 1$

so that $s = 6 - 1 = 5$ & $t = 14 - 1 = 13$

Also $r + u = 75$ (iii)

$r + v = 180$ (iv)

$uv = (7 + r)^2 + 1$ (vi)

Eliminating u & v from (iii) & (iv) gives

$(75 - r)(180 - r) = (7 + r)^2 + 1$

$\Rightarrow 13500 - 255r = 50 + 14r$

$\Rightarrow 269r = 13450$ (C)

Also $qr = 50$ (vii), requiring $r = 50$

As $269 \times 50 = 13450$, (C) is satisfied.

Also $u = 75 - 50 = 25$ & $v = 180 - 50 = 130$

Thus, making the substitutions

$p = 7, q = 1, r = 50, s = 5, t = 13, u = 25, v = 130$

gives the required result.