STEP 2005, Paper 2, Q1 - Solutions (2 pages; 10/5/18)

$\frac{d}{dx}\left(x^2e^{-x^2}\right) = 2xe^{-x^2} + x^2(-2x)e^{-x^2} = 0$

$\Rightarrow xe^{-x^2}(1-x^2) = 0$

$\Rightarrow x = 0, -1\ or\ 1$

For the next part, we want the derivative of $P(x)e^{-x^2}$ to be of the form

$xe^{-x^2}(a^2-x^2)(b^2-x^2)$ [1], in order to give the required solutions

[because e^{-x^2} will clearly be a factor when $P(x)e^{-x^2}$ is differentiated]

Suppose that $P(x) = \sum_{i=1}^{n} c_i x^i$, for some n to be determined

Then $\frac{d}{dx}\left(P(x)e^{-x^2}\right) = (na_nx^{n-1} + \cdots)e^{-x^2} + (a_nx^n + \cdots)(-2x)e^{-x^2}$

e^{-x^2}(-$2a_nx^{n+1} + \cdots)$

Comparing this with [1], we see that n needs to be 4,

and $\frac{d}{dx}\left(P(x)e^{-x^2}\right) = (4a_4x^3 + 3a_3x^2 + 2a_2x + a_1)e^{-x^2}$

$+(a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)(-2x)e^{-x^2}$

Then, equating coefficients of powers of x gives:

x^5: $1 = -2a_4$, so that $a_4 = -\frac{1}{2}$

[there will be no terms in x^4 , $x^2\ or\ x^0$]

x^3: $-a^2 - b^2 = 4a_4 - 2a_2$, so that $a_2 = \frac{1}{2}(a^2+b^2) + 2(-\frac{1}{2})$

ie $a_2 = \frac{1}{2}(a^2 + b^2) - 1$

x: $a^2b^2 = 2a_2 - 2a_0$, so that $a_0 = \frac{1}{2}(a^2 + b^2) - 1 - \frac{1}{2}a^2b^2$

Thus $P(x) = -\frac{1}{2}x^4 + \frac{1}{2}(a^2 + b^2 - 2)x^2 + \frac{1}{2}(a^2 + b^2 - a^2b^2 - 2)$