STEP 2005, Paper 2, Q10 - Solution (2 pages; 11/5/18)

Let the missiles collide at time t from the launch of A.

The position vectors of the two missiles, relative to A, can be equated at time t:

$$\begin{pmatrix} 100cos\alpha.t \\ 100sin\alpha.t - 1/2(10)t^2 \end{pmatrix} = \begin{pmatrix} 180 - 200cos\beta(t-T) \\ 200sin\beta(t-T) - 1/2(10)(t-T)^2 \end{pmatrix}$$

where α & β are the angles of elevation at A & B respectively,

so that $cos\alpha = \frac{4}{5}, sin\alpha = \frac{3}{5}, cos\beta = \frac{3}{5}$ & $sin\beta = \frac{4}{5}$

Thus, $80t = 180 - 120(t-T)$ (1)

& $60t - 5t^2 = 160(t-T) - 5(t-T)^2$ (2)

Then (1) $\Rightarrow 200t = 180 + 120T \Rightarrow t = \frac{9+6T}{10}$ & $t - T = \frac{9-4T}{10}$

Substituting into (2) then gives

$$60\left(\frac{9+6T}{10}\right) - 5\left(\frac{9+6T}{10}\right)^2 = 160\left(\frac{9-4T}{10}\right) - 5\left(\frac{9-4T}{10}\right)^2$$

$\Rightarrow 600(9+6T) - 5(81 + 108T + 36T^2)$

$= 1600(9-4T) - 5(81 - 72T + 16T^2)$

$\Rightarrow 120(9+6T) - (81 + 108T + 36T^2)$

$= 320(9-4T) - (81 - 72T + 16T^2)$

$\Rightarrow T^2(-36+16) + T(720 - 108 + 1280 - 72) + 1080 - 2880 = 0$

$\Rightarrow T^2(-20) + T(2000 - 180) - 1800 = 0$

$\Rightarrow T^2 - T(100 - 9) + 90 = 0$

$\Rightarrow T^2 - 91T + 90 = 0$

$\Rightarrow (T-90)(T-1)=0$

$\Rightarrow T = 90\ or\ 1$

As $t = \frac{9+6T}{10}$, the corresponding values of t are 54.9 and 1.5

Thus $T = 90$ leads to the contradiction that the collision occurs **before** the 2nd missile is launched. When $T = 1$, the collision takes place 0.5 seconds after the 2nd missile is launched, and this is the correct answer. To explain what is going on when $T = 90$, see the diagram below (not to scale), which shows the intersection of two quadratic curves. P corresponds to the situation when $T = 1$ (assuming of course that the missiles are able to meet at P), whilst Q corresponds to $T = 90$: as Q is to the right of P, it represents a time **before** the launch of the 2nd missile.