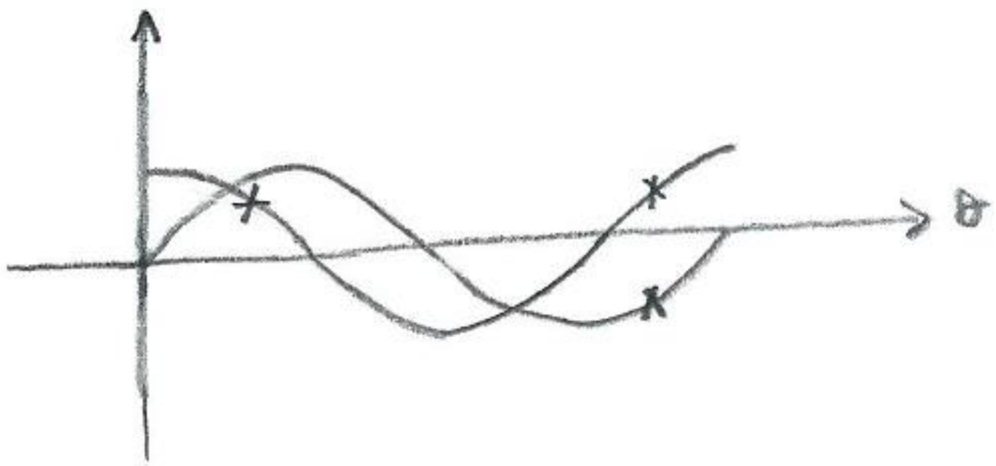
STEP 2005, Paper 1, Q4 – Solution (4 pages; 9/5/18)

(a) The diagram below shows the two values of θ in the range 0 to 2π for which $cos\theta = \frac{3}{5}$ [note that, from the 3,4,5 triangle, the smaller θ *is* $> \frac{\pi}{4}$ (just for the purpose of sketching]. For these two values, $sin\theta = \frac{4}{5}$ & $-\frac{4}{5}$, respectively (from the 3,4,5 triangle; or $cos^2\theta + sin^2\theta = 1$)

So, as $\frac{3\pi}{2} \leq \theta \leq 2\pi$, $sin2\theta = 2sin\theta cos\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}$



$cos3\theta = \cos(2\theta + \theta) = cos2\theta cos\theta - sin2\theta sin\theta$

Now $cos2\theta = cos^2\theta - sin^2\theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$

So $cos3\theta = \left(-\frac{7}{25}\right)\left(\frac{3}{5}\right) - \left(-\frac{24}{25}\right)\left(-\frac{4}{5}\right) = \frac{-21-96}{125} = -\frac{117}{125}$

(b) $tan3\theta = \tan(2\theta + \theta) = \frac{tan2\theta + tan\theta}{1 - tan2\theta tan\theta} = \frac{\frac{2tan\theta}{1-tan^2\theta} + tan\theta}{1 - \frac{2tan\theta}{1-tan^2\theta}.tan\theta}$

$= \frac{2tan\theta+(tan\theta-tan^3\theta)}{(1-tan^2\theta)-2tan^2\theta} = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$

Hence $tan3\theta = \frac{11}{2} \Rightarrow \frac{11}{2} - \frac{33}{2}tan^2\theta = 3tan\theta - tan^3\theta$.

Writing $x = tan\theta$, $f(x)\ (say) = 2x^3 - 33x^2 - 6x + 11 = 0$

We have to find a root $\geq tan\left(\frac{\pi}{4}\right) = 1$

The question is: does this cubic factorise easily?

We can try applying the factor theorem, but unfortunately this doesn't yield any integer roots. At this point you could be forgiven for wondering if a mistake had been made somewhere.

In fact the cubic has a factor of $2x - 1$. However there are a few other possibilities (assuming of course that a convenient factorisation does in fact exist). The full list is:

(1) $(x + 1)(2x^2 + ax + 11)$

(2) $(x - 1)(2x^2 + ax - 11)$

(3) $(x + 11)(2x^2 + ax + 1)$

(4) $(x - 11)(2x^2 + ax - 1)$

(5) $(2x + 1)(x^2 + ax + 11)$

(6) $(2x - 1)(x^2 + ax - 11)$

(7) $(2x + 11)(x^2 + ax + 1)$

(8) $(2x - 11)(x^2 + ax - 1)$

(1) & (2) will have already been eliminated by the Factor theorem. It doesn't take that long to establish whether a consistent value can be found for a in each of the other cases (by equating coefficients), but the process seems a bit pedestrian for a STEP question.

We could attempt a sketch by considering the location of the turning points.

$\frac{dy}{dx} = 6x^2 - 66x - 6$

Then $\frac{dy}{dx} = 0 \Rightarrow x^2 - 11x - 1 = 0$

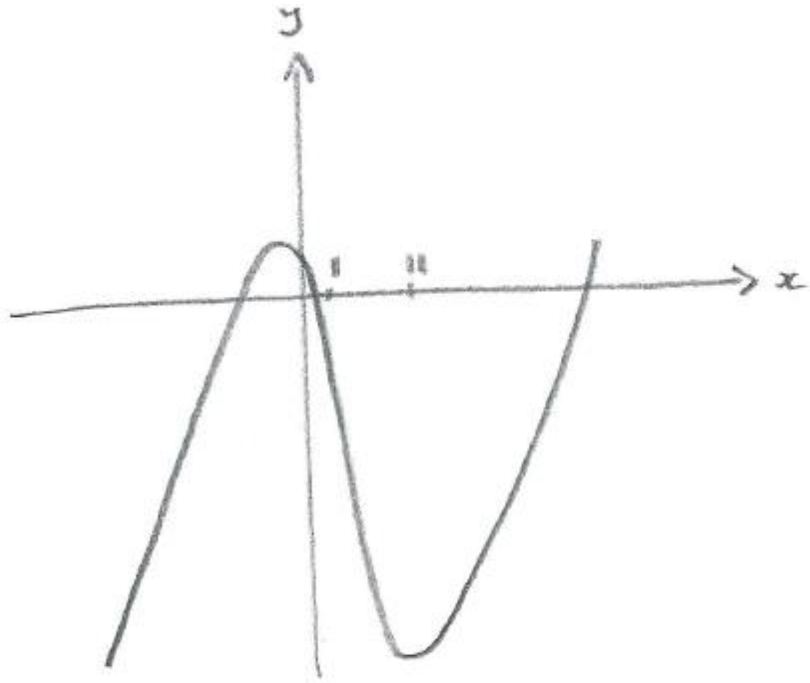
$\Rightarrow x = \frac{11 \pm \sqrt{125}}{2}$; ie just over 11 and just less than 0

Because the coefficient of x^3 is positive, the general shape of $f(x)$ indicates that the maximum will be at just less than 0, whilst the minimim will be at just over 11.

Also, $f(0) = 11$ & $f(1) = -26$, so that there is a root between 0 & 1 (since $f(x)$ is continuous).

Hence we can deduce that there are 3 roots, with the following positions:

< 0 , between 0 & 1, and $>$11



[For any cubic, there will be a point of inflexion halfway between the turning points (if they exist), with the curve having rotational symmetry of order 2 about the point of inflexion.]

From this we can eliminate (4) & (8), but this still leaves 4 options to investigate.

We find that only $2x^3 - 33x^2 - 6x + 11 = (2x - 1)(x^2 + ax - 11)$ gives a consistent value for a:

Equating coefficients of x^2: $-33 = 2a - 1 \Rightarrow a = -16$

Equating coefficients of x: $-6 = -22 - a \Rightarrow a = -16$

So one root is $\frac{1}{2}$ (the one we were expecting between 0 & 1). We are trying to find the root that is ≥ 1, so we want the positive root of $x^2 - 16x - 11 = 0$; namely $\frac{16+\sqrt{16^2+44}}{2} = 8 + \sqrt{64 + 11} = 8 + \sqrt{75}$

[Note that it wasn't necessary to work out 16^2]

[As a check, the negative root, $8 - \sqrt{75}$ is clearly less than $\frac{11-\sqrt{125}}{2}$, the position of the maximum - as expected.]