STEP 2005, Paper 1, Q3 - Solution (2 pages; 9/5/18)

[The phrases "two distinct real solutions" and "exactly one real solution" virtually guarantee that only a quadratic equation is involved - making this question very attractive (especially as all the parts are of the 'show that' type).]

(i) $\frac{x}{x-a}+\frac{x}{x-b}=1 \Rightarrow x(x-b)+x(x-a)=(x-a)(x-b)$

[$x \neq a \; or \; b$ for the original equation to make sense]

$\Rightarrow x^2+x(-b-a+a+b)-ab=0$

$\Rightarrow x^2=ab$

As a & b are either both +ve or both -ve (and real), $ab>0$

Hence the two roots $x=\sqrt{ab}$ & $-\sqrt{ab}$ are real and distinct.

(ii) $\frac{x}{x-a}+\frac{x}{x-b}=1+c \Rightarrow x(x-b)+x(x-a)=(1+c)(x-a)(x-b)$

$\Rightarrow x^2(2-[1+c])+x\left(-b-a+(a+b)(1+c)\right)-(1+c)ab=0$

$\Rightarrow (1-c)x^2+c(a+b)x-(1+c)ab=0$

Exactly one real root $\Leftrightarrow \Delta=0$

[The symbol Δ for the discriminant tends not to be used in A level textbooks (perhaps because it might encourage students to confuse 'discriminant' with 'determinant', which has the same symbol!)]

so that $c^2(a+b)^2+4(1-c)(1+c)ab=0$

$\Leftrightarrow c^2(a+b)^2+4ab(1-c^2)=0$

$\Leftrightarrow c^2(a-b)^2+4ab=0$

$\Leftrightarrow c^2 = \frac{-4ab}{(a-b)^2}$ (1)

$1 - \left(\frac{a+b}{a-b}\right)^2 = \frac{1}{(a-b)^2}((a-b)^2 - (a+b)^2)$

$= \frac{-4ab}{(a-b)^2}$ [as this is a 'show that' result, you might want to deliberately give more working in the exam], so that $c^2 = \frac{-4ab}{(a-b)^2} \Leftrightarrow c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$

As a & b are real, $\left(\frac{a+b}{a-b}\right)^2 \geq 0$, so that $c^2 \leq 1$

Again, as a & b are real, $c^2 \geq 0$

[At this point, we could show very easily that $c^2 > 0$ from $c^2 = \frac{-4ab}{(a-b)^2}$, since a & b are non-zero (2); however, it isn't clear whether the instruction 'deduce' applies to every aspect of the last part. I would have thought that (2) was acceptable, but the Official sol'ns don't do it this way.]

Suppose that $c^2 = 0$, so that $\left(\frac{a+b}{a-b}\right)^2 = 1$

Then $\frac{a+b}{a-b} = 1 \; or -1$,

so that either $a + b = a - b \Rightarrow b = -b \Rightarrow b = 0$ (contradiction)

or $a + b = b - a \Rightarrow a = -a \Rightarrow a = 0$ (contradiction)

Thus $c^2 \neq 0$, and so $0 < c^2 \leq 1$, as required.

[Note that we have only proved that this is a necessary condition for

$c^2 = \frac{-4ab}{(a-b)^2}$ to hold; not (necessarily!) a sufficient condition. There could conceivably be values of c^2 in the range (0,1] for which the result was not true.]