STEP 2005, Paper 1, Q11 – Solution (3 pages; 9/5/18)

[Note that the sketch in (iv) is likely to be fairly time-consuming.]

(i) $sin2t = 2sintcost = 0$ & $2cost = 0 \Rightarrow cost = 0 \Rightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

(ii) [It's probably a good idea to use $\underline{v}$ instead of $\underline{\dot{r}}$, as the dot might get overlooked (either by yourself or the examiner).]

$\underline{v} = 2cos2t\underline{i} - 2sint\underline{j}$

velocity is perpendicular to displacement when $\underline{r}.\underline{v} = 0$, provided that $\underline{r}$ and $\underline{v}$ are not zero [this seems to have been overlooked in the official sol'ns; normally they're quite particular about this sort of refinement]

$\underline{r}.\underline{v} = 0 \Rightarrow 2sin2tcos2t - 4sintcost = 0$

$\Rightarrow sintcostcos2t - sintcost = 0$

$\Rightarrow either\ sint = 0\ \ or\ \ cost = 0\ \ or\ cos2t = 1$

$\Rightarrow t = 0, \pi, \frac{\pi}{2}\ or\ \frac{3\pi}{2}$

but the particle is at the Origin at $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$; ie its displacement vector is $\underline{0}$, so it doesn't make sense to talk about the velocity being perpendicular to it; so we are left with $t = 0\ or\ \ \pi$.

(iii) $\underline{v}$ parallel to $\underline{r}$ $\Rightarrow \frac{2cos2t}{sin2t} = \frac{-2sint}{2cost}$ (1) , provided that $sin2t$ & $cost \neq 0$

As particle is not at O, $cost \neq 0$

If $sin2t = 0$ & $cost \neq 0, then\ sint = 0, so\ that\ t = 0\ or\ \pi$

$t = 0 \Rightarrow \underline{r} = 2\underline{j}$ & $\underline{v} = 2\underline{i}, so\ that\ \underline{r}$ & $\underline{v}\ are\ not\ parallel$

$t = \pi \Rightarrow \underline{r} = -2\underline{j}$ & $\underline{v} = 2\underline{i}, so\ that\ again\ \ \underline{r}$ & $\underline{v}\ are\ not\ parallel$

Then (1) $\Rightarrow 2cos2tcost = -sint(2sintcost)$

$\Rightarrow cos2t + sin^2t = 0\ (as\ cost \neq 0)$

$\Rightarrow (cos^2t - sin^2t) + sin^2t = 0$

$\Rightarrow cos^2t = 0$

$\Rightarrow cost = 0$

But $cost \neq 0$, so that there are no situations when $\underline{v}$ is parallel to $\underline{r}$.

(iv) $|\underline{r}|^2 = sin^2(2t) + 4cos^2t = 4sin^2tcos^2t + 4cos^2t$

$= 4(1 - cos^2t)cos^2t + 4cos^2t$

$= 8cos^2t - 4cos^4t$

$\frac{d}{dt}(8cos^2t - 4cos^4t) = 0$

$\Rightarrow -16costsint + 16cos^3tsint = 0$

$\Rightarrow sint = 0\ (A), cost = 0\ (B)\ or\ cos^2t = 1\ (C)$

$(A) \Rightarrow |\underline{r}|^2 = 4$

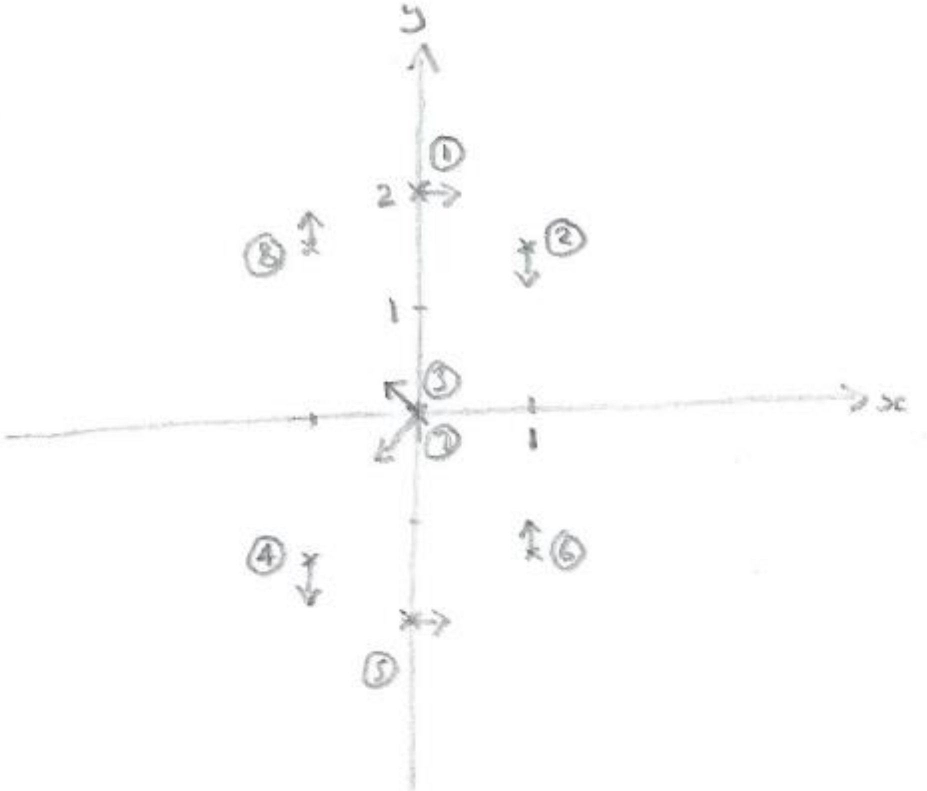
$(B) \Rightarrow |\underline{r}|^2 = 0$

$(C) \Rightarrow t = 0\ or\ \pi$

$t = 0 \Rightarrow |\underline{r}|^2 = 4$ & $t = \pi \Rightarrow |\underline{r}|^2 = 4$ also

So the max. distance is 2

[A fairly safe way of sketching the curve is to establish $\underline{r}$ & $\underline{v}$ at intervals of $\frac{\pi}{4}$ for t, and draw arrows to represent the direction of motion (ie the direction of $\underline{v}$). As it's a bit time-consuming, you might want to save this for the end of the exam, when corners can be cut if necessary.]



[(1) is $t = 0$, (2) is $t = \frac{\pi}{4}$ etc; (9) would be $t = 2\pi$, which is equivalent to $t = 0$; since $cost$ has a period of 2π and $sin2t$ has a period of π (being $sint$ stretched by a scale factor of $\frac{1}{2}$)]