STEP 2005, Paper 3, Q13 - Solution (2 pages; 14/4/21)

(i) 1st part

Only draws resulting in the numbers 0 to w have any effect: we can consider the game to consist of a sequence of events (which must occur eventually) whereby one of these numbers occurs.

To win exactly £3, the first 3 such events must result in a number from 1 to w, with the 4th event resulting in a 0.

2nd part

So P(player wins a total of exactly £3) $= (\frac{w}{w+1})^3\left(\frac{1}{w+1}\right) = \frac{w^3}{(w+1)^4}$, as required.

3rd part

Similarly, P(player wins a total of exactly £r) $= \frac{w^r}{(w+1)^{r+1}}$

4th part

Hence $E(\text{total win}) = \sum_{r=0}^{\infty}\frac{rw^r}{(w+1)^{r+1}}$

Let $\lambda = \frac{1}{w}$; then $E(\text{total win}) = \frac{1+\lambda}{w+1}\sum_{r=0}^{\infty}\frac{r}{(1+\lambda)^{r+1}}$

$= -\frac{1+\lambda}{w+1}\frac{d}{d\lambda}\sum_{r=0}^{\infty}(1+\lambda)^{-r}$

$= -\lambda\,.\frac{w+1}{w+1}\,\frac{d}{d\lambda}\left(\frac{1}{1-\frac{1}{1+\lambda}}\right)$, using the sum to infinity of a Geometric series with common ratio $\frac{1}{1+\lambda} = \frac{w}{w+1} < 1$

$= -\lambda\,\frac{d}{d\lambda}\left(\frac{1+\lambda}{1+\lambda-1}\right) = -\lambda\,\frac{d}{d\lambda}\left(\frac{1}{\lambda}+1\right) = -\lambda(-\lambda^{-2}) = \frac{1}{\lambda} = w$

(ii) Once again, only the numbers 0 to w need to be considered: the pack effectively contains only these numbers.

P(player wins a total of exactly £r)

$$= \frac{w}{w+1} \times \frac{w-1}{w} \times \dots \times \frac{w-(r-1)}{w-(r-2)} \times \frac{1}{w-(r-1)} = \frac{1}{w+1}$$

Then $E(\text{total win}) = \sum_{r=0}^{w} \frac{r}{w+1} = \frac{1}{w+1} \cdot \frac{1}{2} w(w+1) = \frac{w}{2}$, as required.