STEP 2003, Paper 2, Q8 - Solution (4 pages; 12/4/24)

1st Part

By Separation of Variables,

$\int \frac{1}{y}\, dy = -k \int \frac{t^2-3t+2}{t+1}\, dt$,

so that $\ln|y| = -k \int \frac{(t+1)(t-4)+6}{t+1}\, dt$

$= -k \int t - 4 + \frac{6}{t+1}\, dt$

$= -k\left(\frac{1}{2}t^2 - 4t + 6\ln|t+1|\right) - lnC$

(without loss of generality, as $-lnC$ could be any number)

Then $Cy = \exp\left\{-k\left(\frac{1}{2}t^2 - 4t + 6\ln|t+1|\right)\right\}$

When $t = 0, y = A$, so that $C = \frac{1}{A}$,

and hence $y = A\exp\left\{-k\left(\frac{1}{2}t^2 - 4t + 6\ln|t+1|\right)\right\}$

$= A(t+1)^{-6k}\exp\left\{-k\left(\frac{1}{2}t^2 - 4t\right)\right\}$

(It is assumed that $t \geq -1$, so that $(t+1)^{-6k}$ is defined if $k > 0$.)

[t is presumably intended to be time]

2nd Part

A stationary value occurs when $\frac{dy}{dt} = 0$.

Assuming $k \neq 0$ (otherwise $\frac{dy}{dt}$ is always zero),

and noting that $y > 0$ (from (i), as $A > 0$),

$\frac{dy}{dt} = 0$ when $t^2 - 3t + 2 = 0$, assuming that $t \neq -1$

(so that we are now requiring $t > -1$)

ie when $(t-2)(t-1) = 0$,

so that $t = 1\ or\ 2$

The ratio of the two stationary values is

$\frac{(2+1)^{-6k}\exp\{-k\left(\frac{1}{2}2^2-4(2)\right)\}}{(1+1)^{-6k}\exp\{-k\left(\frac{1}{2}1^2-4(1)\right)\}} = (\frac{3}{2})^{-6k}\exp\{-k\left(-6-\frac{1}{2}+4\right)\}$

or, alternatively, taking the reciprocal:

$(\frac{3}{2})^{6k}\exp(-\frac{5}{2}k)$, as required.

(This is the ratio of the value at $t = 1$ to the value at $t = 2$.)

3rd Part

$y = A(t+1)^{-6k}\exp\{-k\left(\frac{1}{2}t^2 - 4t\right)\}$

When $k > 0$, as $t \to \infty$, both $(t+1)^{-6k}$ and $\exp\{-k\left(\frac{1}{2}t^2 - 4t\right)\}$ tend to zero, and so $y \to 0$

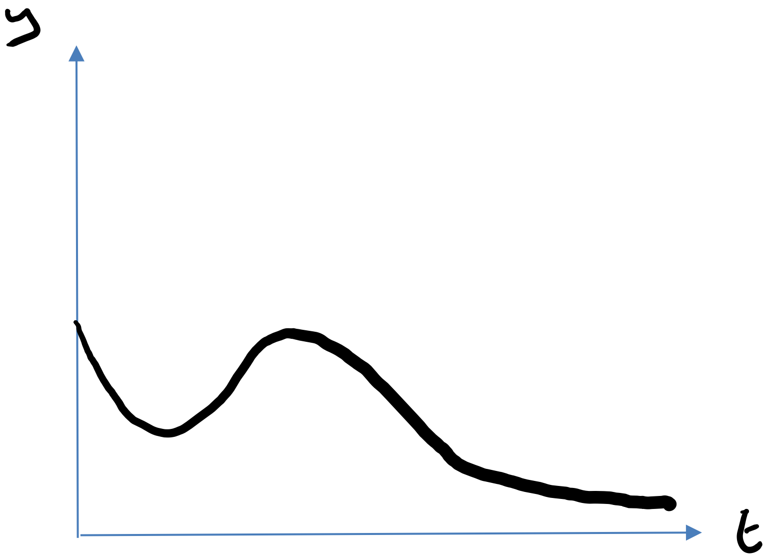
When $k < 0$, $(t+1)^{-6k}$ and $\exp\{-k\left(\frac{1}{2}t^2 - 4t\right)\}$ tend to ∞, and so $y \to \infty$

4th Part

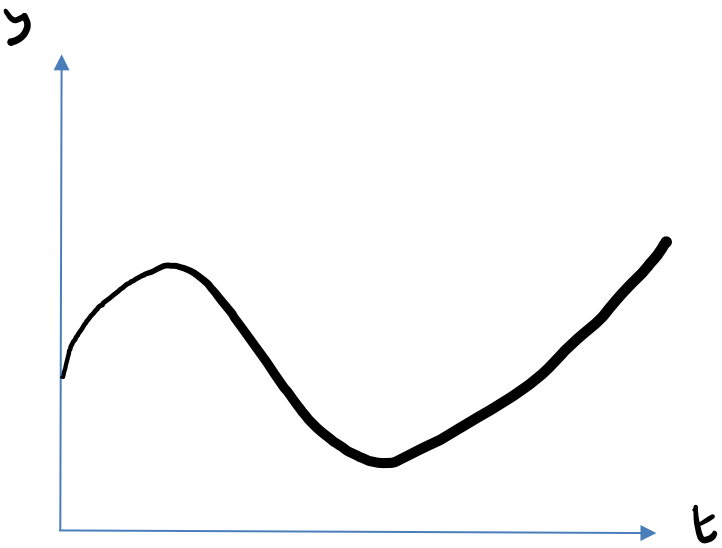
[We could investigate the ratio $(\frac{3}{2})^{6k}\exp\left(-\frac{5}{2}k\right) = (\left(\frac{3}{2}\right)^6 e^{-\frac{5}{2}})^k$.

With a calculator, we can find that $\left(\frac{3}{2}\right)^6 e^{-\frac{5}{2}} < 1$ (it doesn't seem to be that easy to deduce this manually; to 3sf, the value is 0.935), so that when $k > 0$, a minimum occurs at $t = 1$, and a maximum at $t = 2$, and the other way round when $k < 0$. But there is in fact only one way to draw each of the graphs.]

For $k > 0$:



For $k < 0$:



(As a check, the gradient at $t = 0$ can be seen to be negative for $k > 0$, and positive for $k < 0$ – from the original equation.)