STEP 2003, Paper 2, Q7 - Solution (2 pages; 1/4/24)

1st Part

By Parts [differentiating lnx],

$\int_{e^{1/n}}^{\infty} \frac{lnx}{x^{n+1}}\, dx = \left[lnx\, .\, \frac{x^{-n}}{(-n)}\right]_{e^{\frac{1}{n}}}^{\infty} - \int_{e^{1/n}}^{\infty} \frac{1}{x} \cdot \frac{x^{-n}}{(-n)}\, dx$

$= -\frac{1}{n} \lim_{x\to\infty} \frac{lnx}{x^n} + \frac{1}{n}\left(\frac{1}{n}\right) e^{-1} + \frac{1}{n}\int_{e^{1/n}}^{\infty} x^{-n-1}\, dx$

$= \frac{1}{n^2 e} + \frac{1}{n}\left[\frac{x^{-n}}{(-n)}\right]_{e^{1/n}}^{\infty}$, as $\lim_{x\to\infty} \frac{lnx}{x^n} < \lim_{x\to\infty} \frac{lnx}{x} = 0$

$= \frac{1}{n^2 e} + \frac{1}{n}\left(0 + \frac{e^{-1}}{n^2}\right) = \frac{2}{n^2 e}$, as required.

2nd Part

As $1 < a, lnx > 0$ for $x \geq a$, so that $\frac{lnx}{x^{n+1}} > 0$ for $x \geq a$.

Hence $\int_a^{\infty} \frac{lnx}{x^{n+1}}\, dx = \int_a^b \frac{lnx}{x^{n+1}}\, dx + \int_b^{\infty} \frac{lnx}{x^{n+1}}\, dx > \int_b^{\infty} \frac{lnx}{x^{n+1}}\, dx,$

giving the required result.

3rd Part

From the 2nd Part, $\int_{e^{1/n}}^{\infty} \frac{lnx}{x^{n+1}}\, dx < \int_{e^{1/N}}^{\infty} \frac{lnx}{x^{n+1}}\, dx$ for $n < N$

(so that $e^{1/N} < e^{1/n}$)

Then, from the 1st Part, $\frac{2}{n^2 e} < \int_{e^{1/N}}^{\infty} \frac{lnx}{x^{n+1}}\, dx$ for $n < N$,

and so $\frac{1}{n^2} < \frac{e}{2}\int_{e^{1/N}}^{\infty} \frac{lnx}{x^{n+1}}\, dx$ for $n < N$,

and hence $\left(\sum_{n=1}^{N-1} \frac{1}{n^2}\right) + \frac{1}{N^2} < \frac{e}{2}\sum_{n=1}^{N-1}\left(\int_{e^{1/N}}^{\infty} \frac{lnx}{x^{n+1}}\, dx\right) + \frac{1}{N^2}$

(provided that $N > 1$)

$\sum_{n=1}^{N} \frac{1}{n^2} < \frac{e}{2} \int_{e^{1/N}}^{\infty} lnx (\sum_{n=1}^{N-1} \frac{1}{x^{n+1}}) dx + \frac{e}{2} \int_{e^{1/N}}^{\infty} \frac{lnx}{x^{N+1}}\ dx$

(from the 1st Part: $\int_{e^{1/N}}^{\infty} \frac{lnx}{x^{N+1}}\ dx = \frac{2}{N^2 e}$),

and so $\sum_{n=1}^{N} \frac{1}{n^2} < \frac{e}{2} \int_{e^{1/N}}^{\infty} lnx (\sum_{n=1}^{N} \frac{1}{x^{n+1}}) dx$

$= \frac{e}{2} \int_{e^{1/N}}^{\infty} lnx\ (\frac{1}{x^2} (\frac{1-(\frac{1}{x})^N}{1-\frac{1}{x}})) dx$

$= \frac{e}{2} \int_{e^{1/N}}^{\infty} lnx\ (\frac{1-x^{-N}}{x^2 - x}) dx$, as required.