STEP 2003, Paper 2, Q3 - Solution (2 pages; 28/3/24)

[This question seems to be a bit of an aberration, in that it is both very short and very easy (especially for STEP 2) – unless I am missing something!]

1st Part

Let x be an irrational number, and suppose that its cube root is the rational number $\frac{p}{q}$.

Then $(\frac{p}{q})^3 = x$, and therefore $x = \frac{p^3}{q^3}$, which contradicts the assumption that x is irrational. Hence the cube root of x must be irrational.

2nd Part

$u_1 = 5^{\frac{1}{3}}$ or $\sqrt[3]{5}$

Given that $\sqrt[3]{5}$ is irrational, the result (that u_n is irrational) is true for $n = 1$.

Suppose that the result is true for $n = k$, so that $5^{\left(\frac{1}{3^k}\right)}$ is irrational.

Then $u_{k+1} = 5^{\left(\frac{1}{3^{k+1}}\right)} = 5^{\frac{1}{3}\left(\frac{1}{3^k}\right)} = [5^{\left(\frac{1}{3^k}\right)}]^{\frac{1}{3}}$,

which, by the 1st Part, is irrational.

So, if the true is true for $n = k$, then it will be true for $n = k + 1$.

As the result is true for $n = 1$, it is therefore true for $n = 2, 3, \ldots$, and, hence by the principle of induction, it is true for integer

$n \geq 1$.

3rd Part

The sequence $u_n = m.5^{\left(\frac{1}{3^n}\right)}$ [surely] satisfies the requirement [!?]

[Clearly an irrational number multipled by an integer is an irrational number, and this can easily be demonstrated using a proof by contradiction.]