STEP 2003, Paper 2, Q2 - Solution (3 pages; 27/3/24)

1st Part

$\theta = \frac{\pi}{3}$

2nd Part

The result to be proved can be written as

$\frac{\pi}{3} - \arctan\left(\frac{\sqrt{3}}{2}\right) = \arccos\left(\frac{5}{\sqrt{28}}\right)$

From the 1st Part, $4\cos\left(\frac{\pi}{3}\right) + 2\sqrt{3}sin\left(\frac{\pi}{3}\right) = 5$ (*)

Now, if $tan\alpha = \frac{\sqrt{3}}{2}$ for an acute α (so that $\alpha = \arctan\left(\frac{\sqrt{3}}{2}\right)$),

then $cos\alpha = \frac{2}{\sqrt{7}}$ (as $2^2 + (\sqrt{3})^2 = (\sqrt{7})^2$), and $sin\alpha = \frac{\sqrt{3}}{\sqrt{7}}$ (**)

Then (*) $\Rightarrow \frac{4}{\sqrt{28}}\cos\left(\frac{\pi}{3}\right) + \frac{2\sqrt{3}}{\sqrt{28}}sin\left(\frac{\pi}{3}\right) = \frac{5}{\sqrt{28}}$

or $\frac{2}{\sqrt{7}}\cos\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{\sqrt{7}}sin\left(\frac{\pi}{3}\right) = \frac{5}{\sqrt{28}}$

Hence, from (**): $\cos\left(\frac{\pi}{3} - \alpha\right) = \frac{5}{\sqrt{28}}$, where $\alpha = \arctan\left(\frac{\sqrt{3}}{2}\right)$,

and it follows that

$\frac{\pi}{3} - \alpha = \arccos\left(\frac{5}{\sqrt{28}}\right) + 2n\pi$ or $-\arccos\left(\frac{5}{\sqrt{28}}\right) + 2m\pi$

(for integer n & m to be determined)

If $\frac{\pi}{3} = \alpha + \arccos\left(\frac{5}{\sqrt{28}}\right) + 2n\pi$, then n must be 0, as

α & $\arccos\left(\frac{5}{\sqrt{28}}\right)$ are both acute ($n \geq 1$ makes the RHS $> 2\pi$,

whilst $n < 0$ makes the RHS < 0)

And if $\frac{\pi}{3} = \alpha - \arccos\left(\frac{5}{\sqrt{28}}\right) + 2m\pi$,

so that $\frac{\pi}{3} + \arccos\left(\frac{5}{\sqrt{28}}\right) = \alpha + 2m\pi$,

Then in the same way, m must be 0

But $\frac{\pi}{3} - \alpha = \frac{\pi}{3} - \arctan\left(\frac{\sqrt{3}}{2}\right) > \frac{\pi}{3} - \arctan(\sqrt{3}) = \frac{\pi}{3} - \frac{\pi}{3} = 0$,

so that $\frac{\pi}{3} - \alpha > 0$, and therefore m cannot equal 0.

Hence $\frac{\pi}{3} - \arctan\left(\frac{\sqrt{3}}{2}\right) = \arccos\left(\frac{5}{\sqrt{28}}\right)$, as required.

Alternative approach

If $cos\alpha = \frac{5}{\sqrt{28}}$, then $tan\alpha = \frac{\sqrt{28-25}}{5}$

Then show that $tan(\alpha + \beta) = tan(\frac{\pi}{3})$, where $tan\beta = \frac{\sqrt{3}}{2}$,

so that $\alpha + \beta = \frac{\pi}{3} + n\pi$,

and show that $n = 0$

3rd Part

The result to be proved can be written as

$\frac{\pi}{4} + \arctan\left(\frac{3}{4}\right) = \arcsin\left(\frac{7\sqrt{2}}{10}\right)$

Noting that $\arctan\left(\frac{3}{4}\right) = \arcsin\left(\frac{3}{5}\right) = \arccos\left(\frac{4}{5}\right)$,

consider $\sin\left(\frac{\pi}{4} + \arctan\left(\frac{3}{4}\right)\right)$

$= \sin\left(\frac{\pi}{4}\right)\cos\left(\arccos\left(\frac{4}{5}\right)\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$

$= \frac{1}{\sqrt{2}} \cdot \frac{4}{5} + \frac{1}{\sqrt{2}} \cdot \frac{3}{5} = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}$

Hence $\frac{\pi}{4} + \arctan\left(\frac{3}{4}\right) = \arcsin\left(\frac{7\sqrt{2}}{10}\right) + 2n\pi$

or $\pi - \arcsin\left(\frac{7\sqrt{2}}{10}\right) + 2m\pi$

(for integer n & m to be determined)

As $0 < \frac{\pi}{4} + \arctan\left(\frac{3}{4}\right) < \frac{\pi}{4} + \arctan(1) = \frac{\pi}{2}$,

n can only be 0;

and since $\frac{\pi}{2} < \pi - \arcsin\left(\frac{7\sqrt{2}}{10}\right) < \pi$, no m is possible.

And so $\frac{\pi}{4} + \arctan\left(\frac{3}{4}\right) = \arcsin\left(\frac{7\sqrt{2}}{10}\right)$, as required.