STEP 2003, Paper 2, Q1 - Solution (4 pages; 23/4/24)

(i) Consider $\begin{vmatrix} a & -1 & -1 \\ 2a & -1 & -3 \\ 3a & -1 & -5 \end{vmatrix} = a(-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \end{vmatrix}$

$= a\big(2 - 1 + (-1)\big) = 0$

So the eq'ns will never have a unique sol'n.

[A little-known test for whether simultaneous equations are consistent, when they don't have a unique solution, is to replace any one of the columns of the above determinant by the values on the RHS of the eq'ns. If the resulting determinant is zero, then the eq'ns will be consistent. (Compare with the case of a pair of simultaneous eq'ns in two unknowns:

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

Given that $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 0$, so that $\frac{a}{b} = \frac{c}{d}$, $\frac{e}{f} = \frac{a}{b}$ when $\begin{vmatrix} a & e \\ b & f \end{vmatrix} = 0$]

Now, $\begin{vmatrix} 3 & -1 & -1 \\ 7 & -1 & -3 \\ b & -1 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 7 & 1 & 3 \\ b & 1 & 5 \end{vmatrix}$

$= 3(2) - 7(4) + b(2) = -22 + 2b$

So the eq'ns are consistent (for any a) when $-22 + 2b = 0$; ie when $b = 11$]

(In the case $a = 0$),

$-y - z = 3$ (1)

$-y - 3z = 7$ (2)

$-y-5z=b$ (3)

Using (1) to eliminate y,

$(2) \Rightarrow (3+z)-3z=7; z=-2$

& $(3) \Rightarrow (3+z)-5z=b; z=\frac{b-3}{-4}$,

so that the eq'ns are consistent, and there is a sol'n, if and only if

$b=11$

(ii) When $a \neq 0$:

$ax-y-z=3$ (1)

$2ax-y-3z=7$ (2)

$3ax-y-5z=11$ (3)

Using (1) to eliminate terms in ax:

$2(3+y+z)-y-3z=7$ $(2')$

$3(3+y+z)-y-5z=11$ $(3')$,

so that $y-z=1$ & $2y-2z=2$

Then, with $z=\lambda, y=\lambda+1$ & $x=\frac{3+(\lambda+1)+\lambda}{a}$, from (1)

So, as $a \neq 0$, a sol'n exists for any $z=\lambda$

(iii)

$2x-y-z=3$ (1)

$4x-y-3z=7$ (2)

$6x-y-5z=11$ (3)

Using (1) to eliminate y:

$(2) \Rightarrow 4x - (2x - z - 3) - 3z = 7; 2x - 2z = 4$

$(3) \Rightarrow 6x - (2x - z - 3) - 5z = 11; 4x - 4z = 8$

So $z = x - 2$ & $y = 2x - z - 3 = x - 1$

And then $x^2 + y^2 + z^2 = x^2 + (x-1)^2 + (x-2)^2$

$= 3x^2 - 6x + 5$

$= 3(x-1)^2 + 2$

which is minimised when $x = 1, y = x - 1 = 0, z = x - 2 = -1$

(iv)

$ax - y - z = 3$ (1)

$2ax - y - 3z = 7$ (2)

$3ax - y - 5z = b$ (3)

Using (1) to eliminate y:

$(2) \Rightarrow 2ax - (ax - z - 3) - 3z = 7; ax - 2z = 4$

$(3) \Rightarrow 3ax - (ax - z - 3) - 5z = b; 2ax - 4z = b - 3$

$b = 11$, in order for eq'ns to be consistent

Then $z = \frac{a}{2}x - 2$ & $y = ax - z - 3 = \frac{a}{2}x - 1$

And then $y^2 + z^2 = (\frac{a}{2}x - 1)^2 + (\frac{a}{2}x - 2)^2$

$\frac{a^2}{2}x^2 - 3ax + 5$

We require $\frac{a^2}{2}x^2 - 3ax + 5 = y^2 + z^2 < 1$

and $x > 10^6$

Writing $X = ax$, we require $\frac{X^2}{2} - 3X + 4 < 0$;

ie $\frac{1}{2}(X-3)^2 < \frac{1}{2}$, or $(X-3)^2 < 1$,

and hence $-1 < X - 3 < 1$, or $2 < X < 4$

So we require $2 < ax < 4$ with $x > 10^6$

Thus $a < \frac{4}{x} < \frac{4}{10^6}$ and $a > \frac{2}{10^6}$

eg $a = \frac{3}{10^6} = 3 \times 10^{-6}$